

TUW - 96 - 06  
 PITHA - 96/17  
 hep-th/9607226

# Classical and Quantum Gravity in 1+1 Dimensions

## Part III: Solutions of Arbitrary Topology

THOMAS KLÖSCH\*

Institut für Theoretische Physik  
 Technische Universität Wien  
 Wiedner Hauptstr. 8–10, A-1040 Vienna  
 Austria

THOMAS STROBL†

Institut für Theoretische Physik  
 RWTH-Aachen  
 Sommerfeldstr. 26–28, D52056 Aachen  
 Germany

### Abstract

All global solutions of arbitrary topology of the most general 1+1 dimensional dilaton gravity models are obtained. We show that for a generic model there are globally smooth solutions on any non-compact 2-surface. The solution space is parametrized explicitly and the geometrical significance of continuous and discrete labels is elucidated. As a corollary we gain insight into the (in general non-trivial) topology of the reduced phase space.

The classification covers basically all 2D metrics of Lorentzian signature with a (local) Killing symmetry.

PACS numbers: 04.20.Gz 04.60.Kz

*Class. Quantum Grav.* **14** (1997), 1689.

---

\*e-mail: kloesch@tph.tuwien.ac.at

†e-mail: tstrobl@physik.rwth-aachen.de

# 1 Motivation and first results

Much of the interest in two-dimensional gravity models centers around their quantization. However, for any interpretation of quantum results and, even more, for a comparison and possibly an improvement of existing quantization schemes, a sound understanding of the corresponding classical theory is indispensable.

Therefore, in this paper we pursue quite an ambitious goal: Given any 2D gravity Lagrangian of the form [1]<sup>1</sup>

$$L[g, \Phi] = \int_M d^2x \sqrt{|\det g|} [D(\Phi)R - V(\Phi) + Z(\Phi)g^{\mu\nu}\partial_\mu\Phi\partial_\nu\Phi] , \quad (1)$$

we want to classify all its global, diffeomorphism inequivalent classical solutions. This shall be done without any restriction on the topology of the spacetime  $M$ .

For some of the popular, but specific choices of the potentials  $D, V, Z$ , such as those of ordinary (i.e. string inspired, ‘linear’) dilaton gravity, of deSitter gravity, or of spherically reduced gravity, cf. [2], the possible topologies of the maximally extended solutions turn out to be restricted considerably through the field equations. In particular their first homotopy is either trivial or (at most)  $\mathbb{Z}$ . (Allowing e.g. also for conical singularities, cf. Sec. 7, the fundamental group might become more involved.)

For any ‘sufficiently generic’ (as specified below) smooth/analytic choice of  $D, V, Z$ , on the other hand, the field equations of  $L$  allow for maximally extended, globally smooth/analytic solutions on all non-compact two-surfaces with an arbitrary number of handles (genus) and holes ( $\geq 1$ ).<sup>2</sup> This shall be one of the main results of the present paper. These solutions are smooth and maximally extended, more precisely, the boundaries are either at an infinite distance (geodesically complete) or they correspond to true singularities (of the curvature  $R$  and/or the dilaton field  $\Phi$ ). We will call such solutions *global*, as there are other kinds of inextendible solutions (cf. below and Sec. 7).

The existence of solutions on such non-trivial spacetimes is a qualitatively new challenge for any programme of quantizing a gravity theory. Take, e.g., a Hamiltonian approach to quantization: In any dimension  $D+1$  of spacetime the Hamiltonian formulation necessarily is restricted to topologies of the form  $\Sigma \times \mathbb{R}$  where  $\Sigma$  is some (usually spacelike)  $D$ -manifold. In our two-dimensional setting  $\Sigma$  may be  $\mathbb{R}$  or  $S^1$  only. Thus  $\pi_1(M)$  can be  $\mathbb{Z}$  at most. According to our discussion above this is far from exhaustive in most of the models (1). Let us compare this to the case of full four-dimensional Einstein gravity. Clearly, there the space of solutions will include spacetimes of rather complicated topologies. Therefore, a restriction to topologies of the form  $M = \Sigma \times \mathbb{R}$  seems hardly satisfactory in the 4D scenario as well. A path integral approach to quantum gravity, on the other hand, does not place an a priori restriction on the topology of the base manifold

---

<sup>1</sup>In (1)  $g$  is a metric with Lorentzian signature and  $\Phi$  a scalar field, the ‘dilaton field’.  $R$  denotes the Ricci scalar.  $D, V, Z$  are arbitrary (smooth) functions which, for technical reasons, we restrict by  $D' \neq 0$  and either  $Z \neq 0$  or  $Z \equiv 0$ .

<sup>2</sup>That there are no solutions on compact manifolds (except in the flat case, cf. Sec. 6) can be seen by inspection of the possible fundamental groups; however, it may in many cases also be deduced from the fact that the range of the field  $\Phi$  in (1) is not compact. Let us note in this context that according to [3] there are *no* compact two-manifolds without boundary (closed surfaces) that may be endowed with a metric of Lorentzian signature, except for the torus and the Klein bottle.

$M$ . However, also in this approach for  $M \neq \Sigma \times \mathbb{R}$  the definition of an integration measure is plagued by additional ambiguities and problems. The class of models (1) may serve as a good laboratory to improve on that situation and to gain new insights in such directions.

For spacetimes of topology  $\Sigma \times \mathbb{R}$ , furthermore, we are interested in an explicit comparison of the solution space of (1) (space of all solutions to the field equations modulo diffeomorphisms) with the reduced phase space (RPS) in a Hamiltonian formulation of the theory. In the simply connected case ( $\Sigma = \mathbb{R}$ ) we already classified all global diffeomorphism inequivalent solutions in [4]. The solution space was found to be *one*-dimensional, parametrized by a real number  $C \in \mathbb{R}$ . As the result of a symplectic reduction must lead to an *even*-dimensional RPS, we may conclude that in the case of an ‘open universe’ ( $\Sigma = \mathbb{R}$ ) the proper definition of a Hamiltonian system, describing the same physics as (1), is in need of some additional external input. This may creep in implicitly, e.g., when defining boundary/fall-off conditions for the canonical phase space fields or may be introduced by restriction to particular foliations. Periodic boundary conditions, on the other hand, lead to a Hamiltonian formulation which is perfectly well-defined without any further input besides that of periodicity (with respect to some arbitrarily fixed coordinate period). Effectively they describe the case of a ‘closed universe’  $\Sigma = S^1$  and we conclude that for cylindrical topologies of  $M$  the solution space of (1) must be even-dimensional. Indeed it will turn out to be two-dimensional, a second parameter ‘conjugate to  $C$ ’ arising from the ‘compactification’ (in one coordinate), cf. [6].

For generic theories (1) the solution space for  $M \sim S^1 \times \mathbb{R}$ , and thus the corresponding RPS, will have a highly non-trivial topology. This is the second challenge which has to be faced in any quantization scheme: One has to cope with this non-triviality of the orbit space, as for sure the RPS of four-dimensional gravity will be even more intricate.

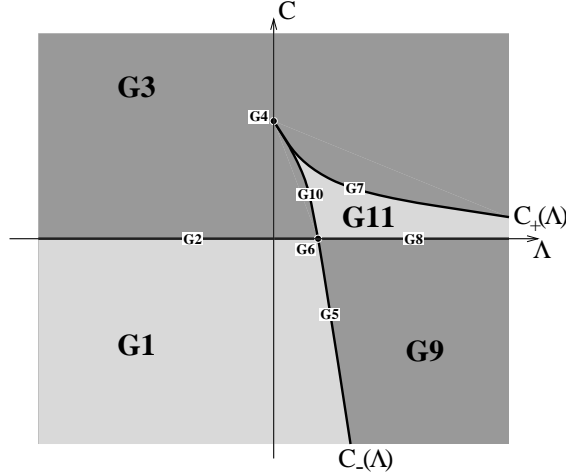
Let us now sketch how to describe the solution space of (1) for arbitrary topologies of  $M$ . Our starting point will be the universal covering solutions, which we determined already in previous papers of this series [2, 4], referred to as **I** and **II**, respectively, in the following: In **I** we showed that for any of the models (1) locally  $g$  may always be brought into a generalized Eddington-Finkelstein form (Eq. (**II**,3)):

$$g = 2dx^0 dx^1 + h(x^0)(dx^1)^2, \quad (2)$$

in which case  $\Phi$  is a function of  $x^0$  only. For the explicit form of these two functions  $h$  and  $\Phi$  we refer the reader to **I**.<sup>3</sup> Here we mention only that, up to diffeomorphisms, they are determined completely in terms of the ‘potentials’  $D, V, Z$ , except for *one* integration constant  $C \in \mathbb{R}$ . As an example, for  $D(\Phi) \equiv \Phi$ ,  $Z(\Phi) \equiv 0$  one obtains  $h = \int^{x^0} V(u) du + C$  and  $\Phi = x^0$ .<sup>4</sup> With formula (2) at hand we can now be more precise about the class of models which allows for all non-compact two-surfaces: This happens whenever the one-parameter family of functions  $h$  of the respective model contains functions  $h$  with three

<sup>3</sup>To generalize  $Z \equiv 1/2$  to an arbitrary  $Z > 0$  just replace  $\rho$  in (**I**,11) by  $\rho = \int^\Phi Z(u)/D'(u) du$ .

<sup>4</sup>Actually, the models considered in **I**, **II** were even more general than those of (1). All of the above holds for generalizations of (1) with non-trivial torsion as well. Also the results have been extended to the case of a (generally dilaton-dependent) coupling of (1) to Yang-Mills fields of an arbitrary gauge group. In the latter case there arise additional parameters labelling the universal covering solutions (cf. **I**, **II**) and certainly the solution space for other topologies changes, too. The content of the present paper may be adapted easily to this more general case, but for simplicity we discuss only models in the absence of Yang-Mills fields.



**Figure 1:** KV-model, survey. The different regions correspond to qualitatively different functions  $h$  (number and degree of zeros, asymptotic behaviour at 0 and  $+\infty$ ). The Penrose diagrams for **G1,3,9,11** are given in Fig. 2.

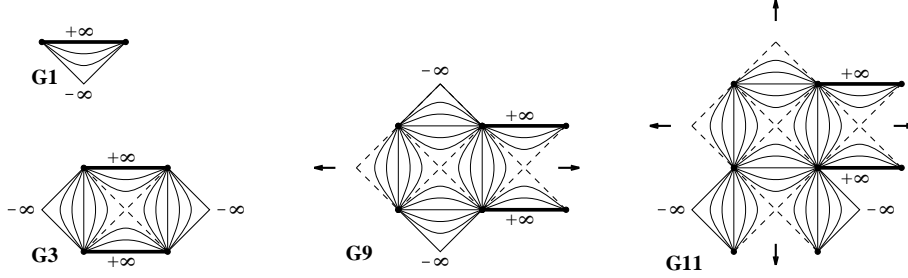
or more zeros. For the above  $h = \int^{x^0} V(u)du + C$ , obviously, this is the case, iff the potential  $V$  changes its sign at least twice. We do not attempt to formulate the analogous conditions on the three potentials  $D, V, Z$  in (1), since in **I** an explicit formula for the one-parameter family  $h$  has been provided and in terms of the latter the condition for non-trivial spacetime topologies is simple enough.

In **II** we constructed the maximal, simply connected extension of the local solution (2). We showed that its global causal structure is determined completely by the (number and kind of) zeros and the asymptotic behaviour of the respective function  $h$ . We derived elementary rules allowing for a straightforward construction of the corresponding Penrose diagrams. As a result one obtains a one-parameter family of universal covering solutions, where the shape of the Penrose diagrams changes with  $C$  only when the function  $h$  changes number and/or degree of its zeros (or its asymptotic behaviour).<sup>5</sup>

As an example we choose a model with torsion, the KV-model [5], the Lagrangian of which consists of three terms: curvature squared, torsion squared, plus a cosmological constant  $\Lambda$ . As this model is well adapted to illustrate much of what has been said up to now, we want to use it in the following to collect first impressions of what to expect when analysing the general model.

In the KV-model the function  $h$  of (2) takes the form  $h = Cx^0 - 2(x^0)^2[(\ln x^0 - 1)^2 + 1 - \Lambda]$ , Eq. (I,60), where  $x^0 \in \mathbb{R}^+$ . Fig. 1 shows a survey of the zeros of  $h$  and its asymptotic behaviour and thus a survey of the various Penrose diagrams. For a negative cosmological constant  $\Lambda$  there are no zeros of  $h$  for  $x^0 \in ]0, \infty[$  if  $C < 0$  and one zero if  $C > 0$ . The respective Penrose diagrams, **G1** and **G3**, are drawn in Fig. 2. Despite some differences

<sup>5</sup>Actually, this is only fully true for the schematic diagrams of **II** (disregarding, e.g., the curved boundaries of **R1** in Figs. **II,11** and **II,7**). However, these changes are irrelevant in the present context, since they do not influence the topology.



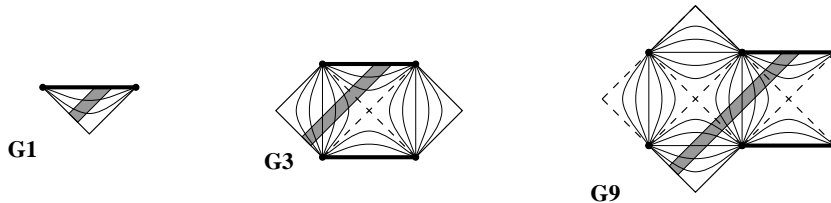
**Figure 2:** Some Penrose-diagrams for the KV-model. The thin lines denote Killing trajectories, the broken lines Killing horizons. The style of the boundary lines indicates their completeness properties; however, since they are irrelevant for our topological considerations, we will treat them rather sloppily. The arrows in **G9** indicate that the patch should be extended horizontally by appending similar copies, and likewise **G11** should be extended vertically and horizontally.

the situation for negative  $\Lambda$  reminds one of spherically symmetric vacuum gravity: Also there a horizon is present only for positive Schwarzschild mass  $m \sim C$ . Moreover, like the spherical model the KV-model with  $\Lambda < 0$  belongs to the class of particular models where the possible topologies of spacetimes are restricted severely. As mentioned above one obtains all non-compact two-surfaces, if the one-parameter family  $h = h_C(x^0)$  contains functions with three or more zeros. In the KV-model this is the case for positive  $\Lambda$ . Correspondingly, there are additional Penrose diagrams then: **G11** if  $\Lambda > 0$  as well as **G9** if  $\Lambda > 1$ , cf. Figs. 1,2.

Let us now discuss the possible global solutions that correspond to the universal covering solutions in Fig. 2. As will be seen later, the results depend only on the number and kind of zeros of the function  $h$ . For reasons of brevity we skip the solutions where  $h$  has higher order zeros in this introductory section; also we postpone a discussion of the deSitter solutions (Sec. 6). They both occur only for positive  $\Lambda$  and if  $C$  takes one of the two particular values  $C_{\pm}(\Lambda)$  on the boundary lines between **G1/G9** or **G3/G11** (i.e. at **G4-7,10** in Fig. 1).

We start with **G1**: Obviously this is a spatially homogeneous spacetime and (2) provides a global chart for it. Identifying  $x^1$  and  $x^1 + \omega$ ,  $\omega = \text{const}$ , evidently we obtain an everywhere smooth solution on a cylindrical spacetime  $M \sim S^1 \times \mathbb{R}$ . It results from the Penrose diagram **G1** by cutting out a (fundamental) region, e.g. the strip between two null-lines in Fig. 3, and gluing both sides together in such a way that the values of the curvature scalar  $R$ , constant along the Killing lines, coincide at the identified ends. The constant  $\omega$  (or a function of it) becomes the variable conjugate to  $C$  here. It may be characterized in an inherently diffeomorphism invariant manner as the (metric induced) distance between two identified points on a line of an arbitrarily fixed value of  $R$  (e.g.  $R = 0$ ). Thus  $\omega$  is a measure for the ‘size of the compact (spacelike) universe’.

Next **G3**: Clearly also in this case we can identify  $x^1$  with  $x^1 + \omega$  in a chart (2); obviously the resulting metric is completely smooth on the cylinder obtained, the fundamental



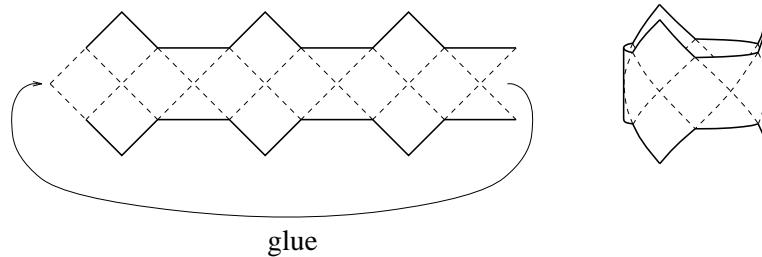
**Figure 3:** Cylinders from **G1,3,9**. The opposite sides of the shaded strip have to be identified along the Killing lines (cf. also Fig. 10). Note that in **G3,9** there occur closed Killing horizons (broken lines), which leads to pathologies of the Taub-NUT type.

region of which is drawn in Fig. 3. However, this cylinder has some pronounced deficiencies: Not only does it contain closed timelike curves as well as one closed null-line (the horizon); this spacetime, although smooth, is geodesically incomplete. There are, e.g., null-lines which wind around the cylinder infinitely often, asymptotically approaching the horizon while having only finite affine length (Taub-NUT spaces, see Sec. 4). So, from a purely gravitational point of view such solutions would be excluded. Having the quantum theory in mind, one might want to regard also such solutions. Being perfectly smooth solutions on a cylinder, certainly they will be contained in the RPS of the Hamiltonian theory. We leave it to the reader to exclude such solutions by hand or not.

The above Taub-NUT solutions are not the only dubious ones. Take any maximally extended spacetime, remove a point from it, and consider the  $n$ -fold covering of this manifold: clearly the resulting spacetime is incomplete. However, for  $n \neq 1$  it is impossible to extend this manifold so as to regain the original spacetime. This is a very trivial example of how to obtain an in some sense maximally extended  $2n$ -kink solution from any spacetime. In the presence of a Killing vector there are, however, more intricate possibilities of constructing kinked spacetimes (resulting, e.g., also in inextendible 2-kink solutions, even flat ones); we postpone their discussion to Sec. 7.<sup>6</sup> These solutions are certainly not global in the sense pointed out before. Excluding them as well as the Taub-NUT type spaces from **G3**, the above solutions are *all* global solutions for the KV-model with negative cosmological constant  $\Lambda$ . In particular we see that the topology of spacetime is planar or cylindrical only. Also the RPS ( $\Sigma = S^1$ ) is found to have a simple structure: It is a plane, parametrized by  $C$  and  $\omega$ .

Now the KV-model with  $\Lambda > 0$ : The discussion of **G1,3** is as above. Also for **G9,11** an identification  $x^1 \sim x^1 + \omega$  in a chart (2) again leads to a smooth (but incomplete) cylinder. However, for **G9,11** there are also cylindrical solutions without any deficiencies. Take, e.g., **G9**: Instead of extending the patch from Fig. 2 infinitely by adding further copies one could take only a finite number of them and glue the faces on the left and the right

<sup>6</sup>One justification for this separation (besides technical issues) is that the RPS of the Hamiltonian formulation introduced in [6] or [7] is in some sense insensitive to these solutions (so effectively one may ignore them to find the RPS). This may be different in other Hamiltonian treatments. We will come back to this issue, cf. [6].



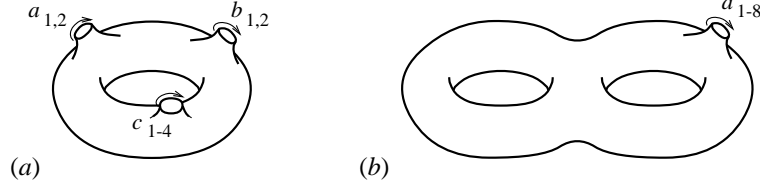
**Figure 4:** Non-Taub-NUT cylinder from **G9**.

side together (cf. Fig. 4). Clearly the result will be an everywhere smooth, inextendible spacetime. Also it allows for a global foliation into  $\Sigma \times \mathbb{R}$  with a spacelike  $\Sigma \sim S^1$ . Now, however, the ‘size’ of the closed universe is ‘quantized’ (the circumference being fully determined by the (integer) number of blocks involved). Still, there is some further ambiguity in the gluing process that leads to a one-parameter family of diffeomorphism inequivalent cylinders for any fixed value of  $C$  and fixed block number. This second quantity, conjugate to  $C$  in a Hamiltonian formulation, and its geometric interpretation shall be provided in the body of the paper (cf. Fig. 14 below).

Thus in the case of **G9** (and similarly of **G11**) we find the solution space for cylindrical spacetimes to be parametrized by  $C$  (within the respective range, cf. Fig. 1), by an additional real gluing parameter, *and* by a further discrete label (block number). For **G9** there is also the possibility of solutions on a Möbius strip: We only have to twist the ends of the horizontal ribbon prior to the identification. It will be shown that these non-orientable solutions are determined uniquely already by fixing  $C$  and the block number; there is now no ambiguity in the gluing!

By far more possibilities arise for **G11**. Again there are cylinders of the above kind, with an analogous parametrization of these solutions. However, now we can also identify faces in vertical direction (cf. Fig. 2). For instance, gluing together the upper and lower ends as well as the right and left ends of the displayed region, one obtains a global solution with the topology of a torus with hole. (It has closed timelike curves, but no further defects; also there are tori without CTCs). The solution space for this topology is three-dimensional now, the two continuous parameters besides  $C$  resulting from inequivalent gluings again. In addition, **G11** allows for much more complicated global solutions. In fact, it is one of the examples for which solutions on all (reasonable) non-compact topologies exist. Fig. 5 displays two further examples: A torus with three holes and a genus two surface with one hole. The respective fundamental regions (for the topologically similar solution **R5**) are displayed in Fig. 17 below. As a general fact the dimension of the solution space exceeds the rank of the respective fundamental group by one, and thus it coincides with twice the genus plus the number of holes. Also, there occur further discrete labels.

In conclusion, let us consider the RPS (= the solution space for topology  $S^1 \times \mathbb{R}$ ) in the case of  $\Lambda > 1$  (simultaneous existence of **G9** and **G11**, cf. Fig. 1). Again it is two-dimensional, being parametrized locally by  $C$  and the respective conjugate ‘gluing’ variable. However, for  $C$  taken from the open interval  $]C_-, C_+[$  (where  $C_{\pm}(\Lambda) =$



**Figure 5:** Factor spaces from **G11**. The corresponding fundamental regions are similar to those given in Fig. 17 for **R5**.

$-4(\pm\sqrt{\Lambda}-1)\exp(\pm\sqrt{\Lambda})$ , cf. Fig. 1 and **II**,37) there are infinitely many such two-dimensional parts of the RPS, labelled by their ‘block number’. More precisely, for sufficiently large negative numbers of the (canonical) variable  $C$  ( $C < C_-$ ) the RPS consists of one two-dimensional sheet. At  $C = C_-$  this sheet splits into infinitely many two-dimensional sheets. At  $C = 0$ , furthermore, any of these sheets splits again into infinitely many, all of which are reunified finally into just one sheet for  $C \geq C_+$ . Furthermore, at  $C_{\pm}$  and 0 the RPS is non-Hausdorff.

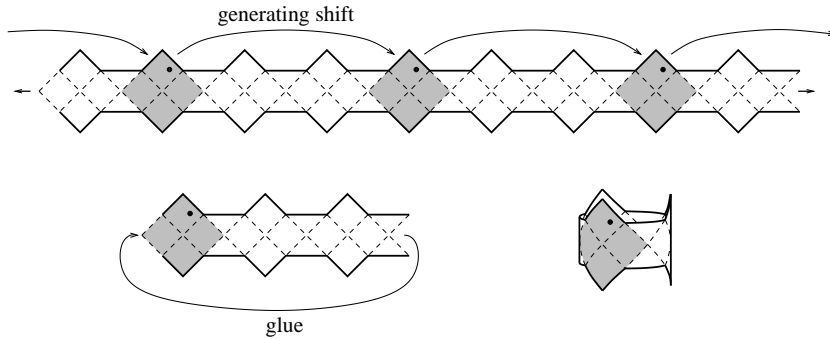
So, while for  $\Lambda < 0$  a RPS quantization is straightforward, yielding wavefunctions  $\Psi(C)$  with the standard inner product, an RPS quantization is not even well-defined for  $\Lambda > 0$  (due to the topological deficiencies of the RPS). In a Dirac approach to quantization [6, 7], on the other hand, related problems are encountered when coming to the issue of an inner product between the physical wave functionals: For  $\Lambda > 1$ , e.g., the states are found to depend on  $C$ , again, but for  $C \in ]C_-, 0[$  there is one further discrete label, and for  $C \in ]0, C_+[$  there are even further labels. When no discrete indices occur, as is the case for  $\Lambda < 0$ , an inner product may be defined by requiring that the Dirac observable  $\omega$  conjugate to  $C$  becomes a hermitean operator when acting on physical states [8]; this again leads to the Lebesgue measure  $dC$  then. Such a simple strategy seems to fail for  $\Lambda > 0$  (and also any generic case of (1)). This is one of the points where an improvement of quantization schemes may set in.

In our above treatment of the KV-model we used a ‘cut-and-paste’ technique to construct the maximally extended solutions from the Penrose diagrams: we cut out some fundamental region from the universal covering solutions and glued it together appropriately. In our classification for the general model (1) a more systematic, group theoretical approach shall be applied. For that purpose we will determine the full isometry group  $\mathcal{G}$  of the universal covering solutions first. This will be achieved in Section 3, after collecting some basics in Section 2. As pointed out there, all global solutions may be obtained as factor solutions of the universal coverings by appropriate discrete subgroups of  $\mathcal{G}$ ; this analysis is carried out in Sections 4,5. However, the simple machinery does not quite apply for those solutions with three Killing vectors, which describe spaces of constant curvature and are thus discussed separately in Section 6. Section 7, finally, treats the non-global inextendible solutions described before.



## 2 Preliminaries

The method employed for finding the multiply connected solutions will be to factor the universal covering solutions by a properly acting transformation group. Let  $\mathcal{G} := \text{Sym}(\mathcal{M})$  be the symmetry group of the manifold  $\mathcal{M}$ . For any subgroup  $\mathcal{H} \leq \mathcal{G}$  we can construct the factor- (or orbit-)space  $\mathcal{M}/\mathcal{H}$  which consists of the orbits  $\mathcal{H}x$  ( $x \in \mathcal{M}$ ) endowed with the quotient topology (e.g. [12]). To pass from this approach to a cut-and-paste description choose a fundamental region, i.e. a subset  $\mathcal{F} \subseteq \mathcal{M}$  such that each orbit  $\mathcal{H}x$  intersects  $\mathcal{F}$  exactly once. The group action then dictates how the points of the boundary of  $\mathcal{F}$  have to be glued together (cf. Fig. 6).



**Figure 6:** Cut-and-paste approach versus factorization. In the upper figure we indicated the action of a transformation group generated by a shift three copies to the right. To obtain the orbit space one identifies all sectors which are a multiple of three copies apart (e.g. all shaded patches). This space may be described equivalently by cutting out a fundamental region (lower figure) and gluing together the corresponding faces.

A priori orbit spaces may be topologically rather unpleasant, they need e.g. not even be Hausdorff. However, iff the action of this subgroup  $\mathcal{H}$  is free and properly discontinuous,<sup>7</sup> then the orbit space is again locally  $\mathbb{R}^n$  and Hausdorff, i.e. a manifold. If, furthermore,  $\mathcal{H}$  preserves some (smooth, metric, etc.) structure or fields (e.g.  $\Phi$ ), then the orbit space inherits such a structure in a unique way, i.e. the metric and the other fields are well-defined on  $\mathcal{M}/\mathcal{H}$  (they ‘factor through’) and still fulfill the equations of motion (e.o.m.).

In this way one can obtain new ‘factor’-solutions of the e.o.m. We now want to sketch shortly that when starting in this way from the universal covering  $\mathcal{M}$ , one obtains *all* multiply connected global solutions: Given a multiply connected manifold  $M$ , one can always construct the (unique) simply connected universal covering space  $\widetilde{M}$  and, furthermore, lift all the structure (metric, fields) to it. Certainly, the lifted fields again satisfy the

<sup>7</sup>‘Free action’ in this context means fixed-point-free (not to be confused with ‘free group’, which means that there are no relations between the generators of the group). For the definition of properly discontinuous see e.g. [10, 11, 12]. These two conditions on the action of  $\mathcal{H}$  certainly imply that  $\mathcal{H}$  is discrete with respect to any reasonable topology on  $\mathcal{G}$ . (The converse is, however, not true: a finite rotation group, e.g., is discrete but has a fixed point).

e.o.m. (since these equations are purely local). Also, the lifted geodesics are geodesics on the covering space, and they have the same completeness properties. Thus the universal covering of a global solution is again a global solution of the e.o.m. and coincides with  $\mathcal{M}$  (which was found to be determined uniquely, cf. **II**). Conversely, the original multiply connected solution  $M$  can be recovered from the universal covering  $\widetilde{M} = \mathcal{M}$  by factoring out the group of deck-transformations. Let us note in passing that the fundamental group of a factor space is isomorphic to the group factored out,  $\pi_1(M) \equiv \pi_1(\mathcal{M}/\mathcal{H}) \cong \mathcal{H}$  (more on this in [12]).

The solutions obtained by this approach are all smooth, maximally extended, and Hausdorff. Of course, if one is less demanding and admits e.g. boundaries (non-maximal extension), conical singularities (failures of the differentiable structure), or violation of the Hausdorff-property, then there are many more solutions. We will shortly touch such possibilities in Sec. 4 (Taub-NUT spaces) and Sec. 7. On the other hand, from the point of view of classical general relativity even the globally smooth solutions may still have unpleasant properties such as closed timelike curves or the lack of global hyperbolicity (cf. the previous section). In any case, our strategy will be to describe all of them; if necessary they may be thrown away afterwards by hand.

Let us shortly summarize what is needed from the first two papers **I** and **II** (while some knowledge of **II** may be useful, a reading of **I** is not necessary): As remarked already in Section 1 the solutions to the model (1) or, more generally, to (**II**,4) could be brought into the Eddington-Finkelstein (EF) form (2) locally, with the dynamical fields  $\Phi$  or  $X^a, X^3$ , respectively, depending on  $x^0$  only. The zeros of  $h$  denote Killing horizons and divide the coordinate patch (2) into *sectors*. A Killing horizon is called non-degenerate, if the corresponding zero of  $h$  is simple ( $h' \neq 0$ ), and degenerate otherwise. The patch was then brought into conformal form, the ‘building block’ (Fig. 7 (b)). Let us note in this context that the shape of the outermost sectors of this building block (square-shaped or triangular) is irrelevant for our analysis.

The metric (2) displays two symmetries, namely the Killing field  $\frac{\partial}{\partial x^1}$ , generating the transformations

$$\tilde{x}^0 = x^0 \quad , \quad \tilde{x}^1 = x^1 + \omega \quad , \quad (3)$$

valid within one building block, and the (local) *flip* transformation

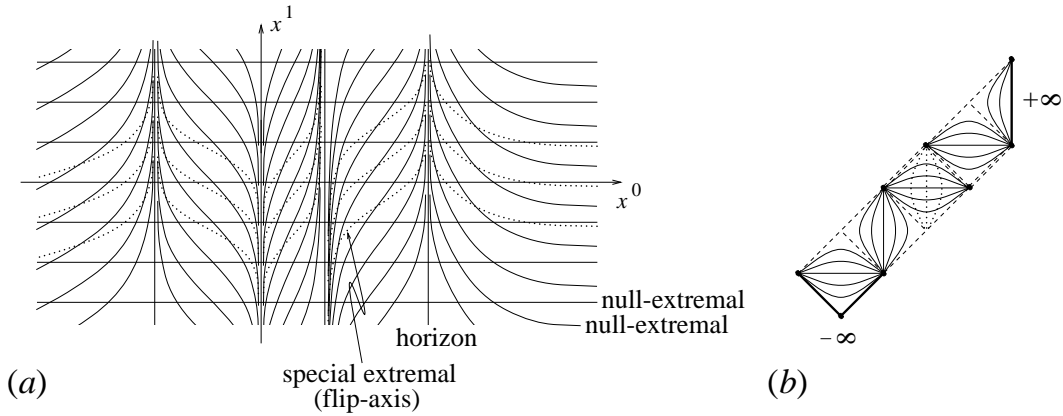
$$\tilde{x}^0 = x^0 \quad , \quad \tilde{x}^1 = -x^1 - 2 \int^{x^0} \frac{du}{h(u)} + \text{const} \quad , \quad (4)$$

valid within a sector (cf. **II**,17), which has been used as gluing diffeomorphism for the maximal extension. The following extremals will be of some interest: the null-extremals (cf. **II**,20–22)

$$x^1 = \text{const} \quad , \quad (5)$$

$$\frac{dx^1}{dx^0} = -\frac{2}{h} \quad , \quad \text{wherever } h(x^0) \neq 0, \quad (6)$$

$$x^0 = \text{const} \quad , \quad \text{if } h(x^0) = 0, \quad (7)$$



**Figure 7:** Building block. (a) EF-coordinates used in Eq. (2), (b) the corresponding (part of the) Penrose diagram. In (a) the null-extremals (5–7) have been drawn, which would run under  $\pm 45^\circ$  in (b). The thin solid lines in (b) are the Killing trajectories,  $x^0 = \text{const}$  (vertical lines in (a)). The broken lines in (b) are Killing horizons (multiple broken ones for degenerate horizons). Finally, the special extremals (8) have been drawn in (a) and in one sector of (b) as dotted lines.

and the special family of non-null extremals (cf. **II**,29)

$$\frac{dx^1}{dx^0} = -\frac{1}{h}, \quad (8)$$

which are also the possible symmetry axes for the flip-transformations (4).

The building block is usually incomplete (unless there is only one sector) and has thus to be extended. This process (described in detail in **II**) consisted of taking at each sector the mirror image of the block and pasting the corresponding sectors together (using the gluing diffeomorphism (4)). Usually, overlapping sectors should not be identified, giving rise to a multi-layered structure, cf. e.g. the spiral-staircase appearance of **G4** (Fig. 11 below). Only where non-degenerate horizons meet in the manner of **G3** (Figs. 2,11, called *bifurcate* Killing horizons), the enclosed vertex point is an interior point (saddle-point for  $\Phi$ , called *bifurcation point*), and the overlapping sectors have to be glued together, yielding one sheet.

Any symmetry-transformation of a solution to the model (1) must of course preserve the function  $\Phi$  (more generally, for the model (**II**,4) the functions  $X^3$ ,  $X^a$ ) and also scalar curvature (and, if non-trivial, also torsion). However, by means of the e.o.m. of (1) the scalar curvature may be expressed in terms of  $\Phi$ . Similarly, for (**II**,4) curvature and torsion can by the e.o.m. (**I**,30) be expressed in terms of the functions  $X^3$  and  $X^a$ . Moreover, since  $X^a$  carries a Lorentz index, one only has to preserve  $(X)^2 \equiv X^a X_a$ , which in turn may be expressed in terms of  $X^3$  via another field equation (Casimir function  $C[(X)^2, X^3] = \text{const}$ , cf. **I**,33,43)). Hence, in order to preserve all the functions above, it

is sufficient to preserve  $X^3$  only.<sup>8</sup> [Recall that, in its specialization to vanishing torsion, (II,4) describes (1) upon the identification  $\Phi = D^{-1}(X^3)$ ; so  $\Phi$  is preserved, iff  $X^3$  is in this case (as common throughout the literature,  $D$  is assumed to be a diffeomorphism). Thus, also for notational simplicity, we shall speak of  $X^3$  only; readers interested merely in (1) may, however, well replace ‘ $X^3$ ’ by ‘ $\Phi$ ’ in everything that follows.]

Finally, we shortly summarize some facts concerning free groups (details can be found in [12, 13]). A free group is a group generated by a number of elements  $g_i$  among which there are no relations. The elements of the group are the words  $g_{i_1}^{k_1} \dots g_{i_l}^{k_l}$ , subject to the relations (necessitated by the group axioms)  $g_i g_i^{-1} = 1$  and  $g_i 1 = 1 g_i = g_i$  (the unit element 1 is the empty word). A word is called *reduced*, if these relations have been applied in order to shorten it wherever possible. Multiplication of group elements is performed simply by concatenating the corresponding words and reducing if necessary. While for a given free group there is no unique choice of the free generators, their number is fixed and is called the *rank* of the group. Free groups are *not* abelian, except for the one-generator group; if the commutation relations  $ab = ba$  are added, then one speaks of a *free abelian* group.

Subgroups of free groups are again free. However, quite contrary to what is known from free abelian groups and vector spaces, the rank of a subgroup of a free group may be larger than that of the original group. The number of the cosets (elements of  $\mathcal{G}/\mathcal{H}$ ) of a subgroup  $\mathcal{H} \leq \mathcal{G}$  is called the *index* of  $\mathcal{H}$  in  $\mathcal{G}$ . If this index is finite, then there is a formula for the rank of the subgroup  $\mathcal{H}$ :

$$\text{index } \mathcal{H} = \frac{\text{rank } \mathcal{H} - 1}{\text{rank } \mathcal{G} - 1}, \quad (9)$$

(cf. [12, 13]). In particular, subgroups of finite index have *never* a smaller rank than the original group. On the other hand,  $\text{rank } \mathcal{H} - 1 = n \cdot (\text{rank } \mathcal{G} - 1)$  for some  $n$  does not guarantee that the index of the subgroup  $\mathcal{H}$  is finite;<sup>9</sup> still, the question of whether a given subgroup has finite index is decidable (cf. [13]), but the algorithm is rather cumbersome.

### 3 The symmetry group

As pointed out in the previous section, any symmetry transformation must preserve the function  $X^3$ ; thus sectors must be mapped as a whole onto corresponding ones (i.e. with the same range of  $X^3$ ). Since  $X^3(x^0)$  is always monotonic (except for the deSitter solutions, which are therefore discussed separately in Sec. 6), this has also the nice consequence that within one building-block a sector cannot be mapped onto another one. So each transformation gives rise to a certain permutation of the sectors and we can thus split it

---

<sup>8</sup>One could also consider neglecting preservation of  $\Phi$  resp.  $X^i$  and regard isometries only, e.g. when one is interested merely in a classification of all global 1 + 1 metrics with one (local) Killing field. In cases where  $R(X^3)$  is not one-to-one this may lead to further discrete symmetries. We will not discuss the additional factor spaces that can arise as a consequence.

<sup>9</sup>For instance, there are a lot of proper subgroups  $\mathcal{H}$  with the same rank as  $\mathcal{G}$  (e.g. those generated by powers of the original free generators). However, none of them can be of finite index: If they were, then according to (9) they should have index 1; but this means that there is no coset besides  $\mathcal{H}$ , thus  $\mathcal{H} = \mathcal{G}$ , contrary to the assumption.

into (i) a sector-permutation and (ii) an isometry of a sector onto itself. Furthermore, it is evident that the whole transformation is already fully determined by the image of only one sector (the transformation can then be extended to the other sectors by applying the gluing diffeomorphism (4)).

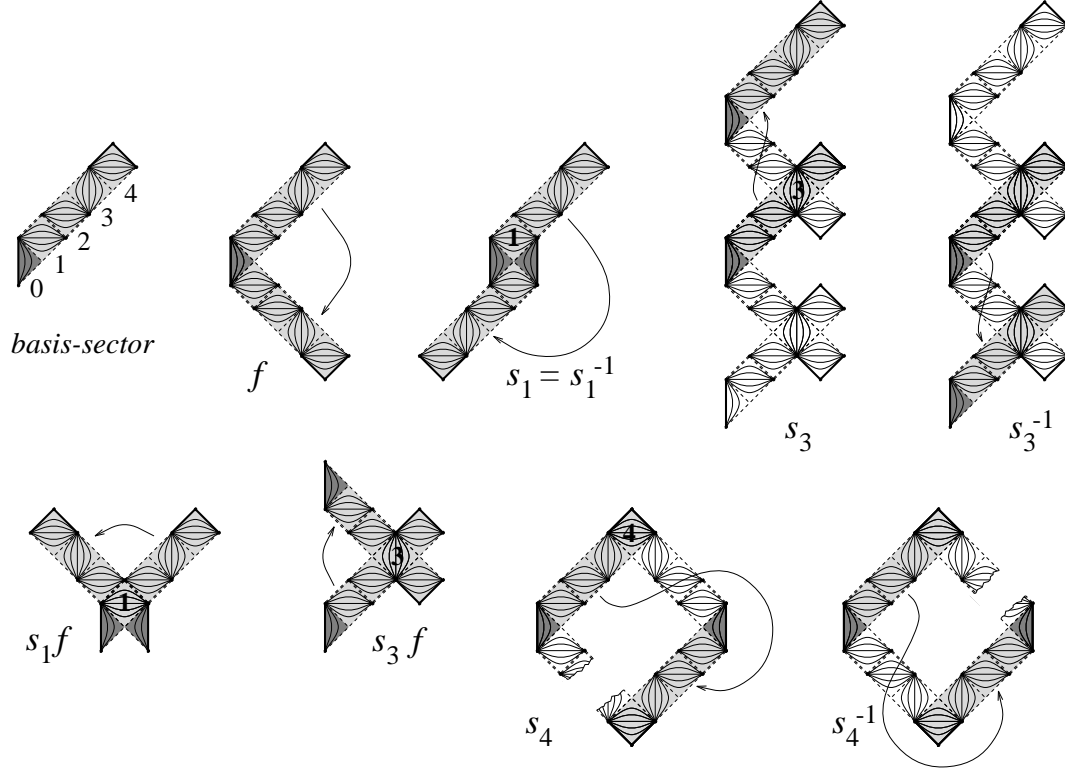
Let us start with (ii), i.e. determine all isometries of one sector onto itself. Again  $X^3$  must be preserved, so in the chart (2) the map must preserve the lines  $X^3 = \text{const} \Leftrightarrow x^0 = \text{const}$ . But also null-extremals must be mapped onto null-extremals. This leaves two possibilities: If the null-extremals (5) (i.e.  $x^1 = \text{const}$ ) are mapped onto themselves, then the only possibility is an overall shift of the  $x^1$ -coordinate,  $x^1 \rightarrow x^1 + \omega$ , i.e. a Killing-transformation (3). The gluing diffeomorphism (4) shows that such a transformation extends uniquely onto the whole universal covering, and that in all charts (2) it is represented as an  $x^1$ -shift of the same amount (but on the ‘flipped’ ones in the opposite direction!). In the neighbourhood of bifurcation points (simple zero of  $h(x^0)$ ) we can also use the local Kruskal-coordinates (II,33), where the same transformation reads  $u \rightarrow u \exp(\frac{h'(a)}{2}\omega)$ ,  $v \rightarrow v \exp(-\frac{h'(a)}{2}\omega)$ , which looks in this case very much like a Lorentz-boost (cf. e.g. the arrows around the bifurcation points in Figs. 14, 20 below). We will (thus) call these Killing-transformations shortly *boosts* and denote them by  $b_\omega$ . The composition law is clearly  $b_\omega b_\nu = b_{\omega+\nu}$ , so the boosts form a group isomorphic to  $\mathbb{R}$ , which shall henceforth be denoted by  $\mathbb{R}^{(\text{boost})}$ .

If, on the other hand, the two families of null-extremals, (5) (i.e.  $x^1 = \text{const}$ ) and (6) (i.e.  $dx^1/dx^0 = -2/h$ ), are interchanged, then we obtain precisely the flip transformations. These transformations are the gluing diffeomorphisms (4), but this time considered as active transformations. Due to the constant in (4) the flips come as a one-parameter family; however, they differ only by a boost, i.e. given a fixed flip transformation  $f$ , any other flip  $f'$  can be obtained as  $f' = f b_\omega \equiv b_{-\omega} f \equiv b_{\omega/2}^{-1} f b_{\omega/2}$  for some  $\omega$ . We will thus consider only *one* flip and denote it by  $f$ . In the Penrose diagrams such a flip is essentially a reflexion at some axis (horizontal or vertical, according to  $\text{sgn } h$ ; cf. Fig. 8) and it is of course involutive (i.e. self-inverse,  $f^2 = 1$ ). Let us finally point out that while under a pure boost each sector is mapped onto itself (the corresponding sector-permutation is the identity), a flip transformation always (unless there is only one sector) entails a non-trivial sector-permutation (which is clearly self-inverse, since  $f$  is).

We now turn to task (i), the description of the sector-permutations: As pointed out above, each transformation is fully determined by its action on only one sector. Let us thus choose such a ‘basis-sector’. If the transformation does not preserve orientation,<sup>10</sup> we may apply a flip at this basis-sector and we are left with an orientation-preserving transformation. For these, however, the corresponding sector-permutation is already uniquely determined by the image of the basis-sector, i.e., for each copy of the basis-sector somewhere in the universal covering there is exactly one sector-permutation moving the basis-sector onto that copy. All these orientation-preserving sector-permutations thus make up a discrete combinatorial group (henceforth called {sector-moves}), which now shall be described in more detail.

Choose also a ‘basis-building-block’ and within this basis-block fix the first (or better: zeroth) sector as basis-sector (we label the sectors by  $0, \dots, n$  and the horizons between

<sup>10</sup>By this we mean the orientation of the spacetime considered as a 2-manifold, not the *spatial* orientation.



**Figure 8:** Some basic sector-moves, their inverses, and flips. The light shaded block is the basis-block resp. its image under the move, the dark shaded sector is the basis-sector (or its image). Only a few sectors of the universal covering are displayed. Note that the basic move  $s_4$  and its inverse  $s_4^{-1}$  transport the basis-block into different (but overlapping) layers of the universal covering. And note also that  $s_1$  (a bifurcation point reflexion),  $s_4$ , and their inverses turn the basis-block upside down!

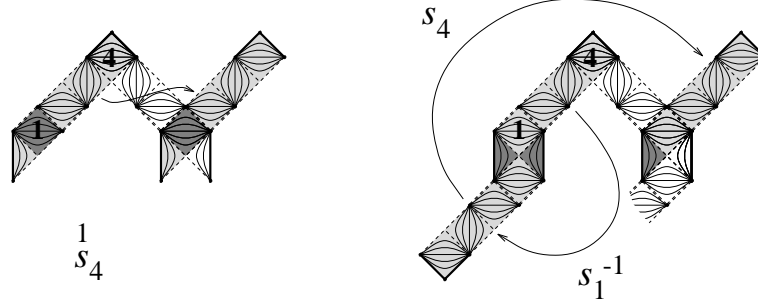
by  $1, \dots, n$ ). By a ‘basic move’ across sector  $i$  we mean the following (cf. Fig. 8): Go from the basis-sector to the  $i$ th sector of the basis-block and from there to the zeroth sector of the perpendicular (i.e. flipped) block. The corresponding sector-move, mapping the basis-sector onto this other sector, shall be denoted by  $s_i$  (of course  $s_0$  is the identity,  $s_0 = 1$ ). Also the inverse basic moves can be easily described: One has to perform the same procedure at the *flipped* basis-block only (cf. Fig. 8). An inverse basic move is thus the conjugate of the original move by a flip,  $f s_i f = s_i^{-1}$  (note  $f = f^{-1}$ ). Here  $f$  has been supposed to be a flip at the basis-sector. Flips at other sectors can be obtained by composition with sector-moves:  $s_i f = f s_i^{-1}$  is a flip at sector  $i$  of the basis-block (Fig. 8).

Evidently there are two qualitatively different cases: If basis-sector and ‘flip’-sector are both stationary resp. spatially homogeneous, then the basic move is essentially a translation (e.g.  $s_3$  in Fig. 8). However, if the sectors are of a different kind, then the move ( $s_1, s_2, s_4$  in the example of Fig. 8) turns the whole solution upside down, inverting space *and* time (thus still preserving the orientation, as required for elements of  $\{\text{sector-moves}\}$ ).

— in contrast to, e.g.,  $s_1 f$ , which inverts space only, cf. Fig. 8).

[Of course it is not necessary to choose the zeroth sector of the block as basis-sector.

Let us denote the basic move across sector  $i$  with basis-sector  $k$  by  $s_i^k$  (thus  $s_i^0 \equiv s_i$ ). They can, however, be expressed in terms of the old moves: As may be seen from Fig. 9 we have  $s_i^k = s_i s_k^{-1}$ , and consequently even  $s_i^k = s_i^l s_k^{-1}$  for an arbitrary sector  $l$ . Obviously always  $s_k^k = 1$  (generalizing  $s_0 = 1$ ) and  $s_i^k = s_k^{-1}$ . Thus, there is no loss of generality in choosing the basis-sector zero.]



**Figure 9:** Basic sector-move with different basis-sector. As is seen from this figure the move  $s_4^1$  can be composed of two moves with basis-sector zero,  $s_4^1 = s_4 s_1^{-1}$ . Although in this case  $s_1^{-1} = s_1$  (relation at a bifurcation point), in general the right element has to be an inverse move!

The above basic moves  $s_i$  form already a complete set of generators for {sector-moves}: By the extension process each location in the universal covering can be reached from the basis-sector by (repeated) application of the following step: Move through the basis-block or the flipped basis-block to some sector, then continue along the perpendicular block (i.e., applying a flip at this sector). But this step *is* exactly a basic move.

There may yet be relations between the generators. If, however, all horizons are degenerate, then there are no relations, and the resulting group is the free group with generators  $s_i$ . The reason is that in this case the vertex points between sectors are at an infinite distance (cf. II,30) and thus the overlapping sectors (after surrounding such points) must not be identified, yielding a multilayered structure (cf. e.g.  $s_4$  in Fig. 8, or the ‘winding staircase’-like **G4** in Fig. 11 below). A non-trivial word composed of basic moves  $s_i$  describes such a move, which must therefore necessarily lead into a different ‘layer’ of the universal covering, since this manifold is simply connected.

The situation is different if there are non-degenerate horizons: In this case there are (interior!) bifurcation points at which four sectors meet, which then constitute a single sheet. Moving once around such a bifurcation point leads back to the original sector. Consequently there emerges a relation: If e.g. the first horizon is non-degenerate then the basic move  $s_1$  turns the solution  $180^\circ$  around with the bifurcation point as centre (cf. Fig. 8). A second application of  $s_1$  yields the original configuration again, so we have  $s_1^2 = 1$ . To find the analogous relations for a non-degenerate horizon say between the  $(k-1)$ th and  $k$ th sector it is wise to temporarily shift the basis-sector to the  $k$ th sector.

Then evidently  $s_{k-1}^k$  turns the solution  $180^\circ$  around that bifurcation point and we have the relation  $(s_{k-1}^k)^2 = 1$  which translates back to  $(s_{k-1}s_k^{-1})^2 = 1$ .

Summarizing, we have found that the group {sector-moves} has the following presentation in terms of generators and relations ( $n$  being the number of horizons):

$$\{\text{sector-moves}\} = \left\langle s_1, \dots, s_n \mid (s_{k-1}s_k^{-1})^2 = 1 \text{ for each non-degenerate horizon } k \right\rangle. \quad (10)$$

Any symmetry transformation can thus be written as a product of possibly a flip  $f$  (if it is orientation-reversing), a sector-move from the group (10), and a boost  $b_\omega$  ( $\omega \in \mathbb{R}$ ). Furthermore, this representation is unique, provided the elements are in this order. We have yet to describe the group product: Evidently boosts and sector-moves commute, since, as pointed out previously, a boost is presented in all equally oriented charts (and sector-moves preserve the orientation) as a shift  $x^1 \rightarrow x^1 + \omega$ , Eq. (3). Furthermore, the conjugate of a boost by a flip is the inverse boost,  $fb_\omega f = b_{-\omega} = b_\omega^{-1}$ , and the conjugate of a basic move by a flip is the inverse basic move,  $fs_i f = s_i^{-1}$ , which also defines the conjugate of a general (composite) sector-move. The group product is thus completely determined, since in the formal product of two elements the factors can be interchanged to yield the above format. Thus the structure of the full isometry group is a semi-direct product

$$\mathcal{G} = \mathbb{Z}_2^{(\text{flip})} \ltimes \left( \mathbb{R}^{(\text{boost})} \times \{\text{sector-moves}\} \right), \quad (11)$$

where the right factor (in round brackets) is the normal subgroup,  $\mathbb{Z}_2^{(\text{flip})}$  denotes the group  $\{1, f\}$ , and  $\mathbb{R}^{(\text{boost})}$  is the group of boosts described previously. In particular, if we restrict ourselves to orientable factor spaces and thus to orientation-preserving isometries, flips must be omitted and we are left with a *direct* product of the combinatorial group {sector-moves} with  $\mathbb{R}^{(\text{boost})}$ .

Still, this description is not always optimal, for two reasons: First, one is often interested in orientable *and* time-orientable solutions. Second, while for only degenerate horizons the group {sector-moves} is a free group, this is not true if there are non-degenerate horizons (cf. Eq. (10)). Interestingly, both problems can be dealt with simultaneously. Clearly the time-orientation-preserving sector-moves constitute a subgroup of {sector-moves}, which we shall denote by  $\{\text{sector-moves}\}^\uparrow$  (in analogy to the notation used frequently for the orthochronous Lorentz group). If all horizons are degenerate and of even degree, then all sectors are equally ‘oriented’ (stationary or spatially homogeneous) and consequently all sector-moves are automatically time-orientation-preserving; thus  $\{\text{sector-moves}\}^\uparrow = \{\text{sector-moves}\}$ , and its rank equals the number of horizons.

This is no longer the case if there are horizons of odd degree. Then some sectors will have an orientation contrary to that of the basis-sector and a sector-move at such a sector will reverse the time-orientation (cf., e.g.,  $s_1$  in Fig. 8). The group  $\{\text{sector-moves}\}^\uparrow$  is then a proper subgroup of {sector-moves} (with index 2), which consists of all elements containing an even number of time-orientation reversing sector-moves.

Fortunately, this group can be described quite explicitly. Let us start with the case that there are non-degenerate horizons. To simplify notation we assume for the moment



the first horizon to be non-degenerate (below we will drop this requirement). The sector-move  $s_1$  is then an reflexion at a bifurcation point and thus inverts the time-orientation (cf. Fig. 8). But also the sector-moves around all other sectors ‘oriented’ contrary to the basis-sector (sectors 2 and 4 in Fig. 8) will reverse the time-orientation. The idea is to extract the space-time-inversion  $s_1$  from the group {sector-moves}: For each sector  $i$  introduce the new generators  $s_i$  and  $s_1 s_i s_1$ , if the  $i$ th sector has the same ‘orientation’ as the basis-sector, and  $s_i s_1$  and  $s_1 s_i$ , if it has the opposite ‘orientation’. These new generators are all time-orientation-preserving. Together with  $s_1$  they obviously still span the whole group {sector-moves} (relations between the new generators will be discussed presently). Conjugation by  $s_1$  only permutes them among themselves (use  $s_1^2 = 1$ ):  $s_i s_1 \leftrightarrow s_1 s_i$  and  $s_i \leftrightarrow s_1 s_i s_1$ . Also, any element of {sector-moves} can be expressed as a word composed of the new generators with or without a leading  $s_1$ . Thus the group is a semidirect product

$$\{\text{sector-moves}\} = \mathbb{Z}_2^{(\text{PT})} \ltimes \{\text{sector-moves}\}^\uparrow, \quad (12)$$

where  $\mathbb{Z}_2^{(\text{PT})} = \{1, s_1\}$  (the ‘PT’ stands for parity and time-inversion) and  $\{\text{sector-moves}\}^\uparrow$  is the group generated by the new elements given above (with  $i \geq 2$ ). There may still be some relations among these generators. From (10) we know that any non-degenerate horizon  $k$  adds a relation  $(s_{k-1} s_k^{-1})^2 = 1$  or equivalently  $s_{k-1} s_k^{-1} = s_k s_{k-1}^{-1}$ . This yields a relation between the new generators, e.g.  $(s_1 s_{k-1} s_1)(s_k s_1)^{-1} = (s_1 s_k) s_{k-1}^{-1}$ , by means of which either of the (four) generators involved can be expressed in terms of the remaining ones. Apart from those relations there are no further ones, so if we eliminate the redundant generators we are left with a *free* group! To determine the rank of this group note that each sector  $> 1$  contributes two generators and each non-degenerate horizon (except for the first, which was taken into account already in  $\mathbb{Z}_2^{(\text{PT})}$ ) yields a relation which in turn kills one generator. Thus

$$\begin{aligned} \text{rank } \{\text{sector-moves}\}^\uparrow &= 2 (\text{number of degenerate horizons}) + \\ &\quad + \text{number of non-degenerate horizons} - 1. \end{aligned} \quad (13)$$

Finally, if the first horizon is degenerate but the  $k$ th is not, then one can replace  $s_1$  above by  $s_{k-1}^k$  and proceed in an analogous way. The result is (12,13) again.<sup>11</sup>

---

<sup>11</sup>Of course there may be other possibilities to split the isometry group into a product. For instance, if there are bifurcate horizons, then one can choose a basis-bifurcation-point and instead of tracking the motion of the basis-sector (which leads to {sector-moves}) follow the bifurcation point. The resulting group of space- and time-orientation-preserving bifurcation point moves coincides exactly with  $\{\text{sector-moves}\}^\uparrow$ , which can thus also be interpreted as {bifurcation point moves} (cf. e.g. Figs. 13, 15 (b), 16). The remaining bifurcation point preserving isometries make up the 1+1 dimensional Lorentz group  $O(1, 1)$ , which contains the boosts, flips (i.e. space-inversions and time-inversions), and reflexion at the bifurcation point (i.e. space-time-inversion). Hence the full isometry group can also be written as

$$\mathcal{G} = O(1, 1) \ltimes \{\text{bifurcation point moves}\}.$$

Furthermore, when restricting to space- and time-orientable solutions we may use

$$\mathcal{G}_+^\uparrow = SO^\uparrow(1, 1) \ltimes \{\text{bifurcation point moves}\}$$

where the proper orthochronous Lorentz group  $SO^\uparrow(1, 1) \cong \mathbb{R}^{(\text{boost})}$  contains only the boosts. Clearly, this is just the same as Eq. (14).

The remaining case to consider is the one in which all the horizons are degenerate. Then  $\{\text{sector-moves}\}$  (10) is a free group and correspondingly its subgroup  $\{\text{sector-moves}\}^\uparrow$  is free, too (cf. Sec. 2). Let us determine its rank: As noted already above, if all the horizons are of even degree, then  $\{\text{sector-moves}\}^\uparrow = \{\text{sector-moves}\}$ , and the rank equals the total number of (degenerate) horizons (thus (13) does not generalize to this case). Finally, assume that there is an odd-degree horizon and let it again be the first one (if it is not the first but the  $k$ th, just replace  $s_1$  by  $s_{k-1}^k$  in what follows): A set of generators can be found in the same way as above (introducing  $s_1 s_k$ ,  $s_k s_1$ , or  $s_l$ ,  $s_1 s_l s_1$ , respectively,  $k, l \geq 2$ ); however, now the element  $s_1^2$ , which no longer is the unit element, has to be added as a further generator. If  $d$  denotes the number of (degenerate) horizons, we thus find  $2d - 1$  generators for  $\{\text{sector-moves}\}^\uparrow$  (two for each degenerate horizon except for the first horizon, which adds one only). Here we also could have used formula (9), since  $\{\text{sector-moves}\}^\uparrow$  has index 2 in  $\{\text{sector-moves}\}$ .<sup>12</sup>

Let us finally summarize the above results:

**Theorem:** The group of space- and time-orientation-preserving symmetry transformations  $\mathcal{G}_+^\uparrow$  is a direct product of  $\mathbb{R}$  with a free group,

$$\mathcal{G}_+^\uparrow = \mathbb{R}^{(\text{boost})} \times \{\text{sector-moves}\}^\uparrow. \quad (14)$$

Let furthermore  $n$  denote the number of non-degenerate horizons and  $d$  the number of degenerate horizons. Then the rank of the free group  $\{\text{sector-moves}\}^\uparrow$  is

$$\begin{array}{ll} d & \text{if all horizons are of even degree,} \\ 2d + n - 1 & \text{in all other cases.} \end{array} \quad (15)$$

We will mainly work with this subgroup, but nevertheless discuss at a few examples how (time-)orientation-violating transformations can be taken into account.

## 4 Subgroups and factor spaces

We now come to the classification of all factor solutions. As pointed out in Section 2, they are obtained by factoring out a freely and properly discontinuously acting (from now on called shorthand *properly acting*) symmetry group from the universal covering solutions. Thus we have first to find all properly acting subgroups of the full symmetry group. However, not all different subgroups give rise to different (i.e. non-isometric) factor spaces. If, for instance, two subgroups  $\mathcal{H}$  and  $\mathcal{H}'$  are conjugate (i.e., there is a symmetry transformation  $h \in \mathcal{G}$  such that  $\mathcal{H}' = h\mathcal{H}h^{-1}$ ) then the factor spaces  $\mathcal{M}/\mathcal{H}$  and  $\mathcal{M}/\mathcal{H}'$  are isometric. (Roughly speaking, this conjugation can be interpreted as a coordinate change). But also the converse is true:

---

<sup>12</sup>Note, however, that in this case  $\{\text{sector-moves}\}$  does not split into a semi-direct product like in (12), since now there is no subgroup  $\mathbb{Z}_2^{(\text{PT})}$  ( $\{\text{sector-moves}\}$  is free!); it is only a non-trivial extension of  $\mathbb{Z}_2$  with  $\{\text{sector-moves}\}^\uparrow$ .

**Theorem:** Two factor spaces are isometric iff the corresponding subgroups are conjugate.

(for a proof cf. e.g. [11], Lemma 2.5.6).

The possible factor spaces are thus in one-to-one correspondence with the conjugacy classes of properly acting subgroups. Still, apart from this abstract characterization it would be nice to have some ‘physical observable’, capable of discerning between the different factor spaces. This concerns mainly the boost-components of the subgroup elements, since information about the sector-moves is in a rather obvious way encoded in the global causal structure (number and arrangement of the sectors and singularities) of the factor space. Indeed we will in general be able to find such observables. [Here ‘physical observable’ means some quantity that remains unchanged under the group of diffeomorphisms (= gauge symmetry); so it will be a *geometrical invariant*, which not necessarily can be ‘measured’ also by a ‘physical observer’, strolling along his timelike worldline].

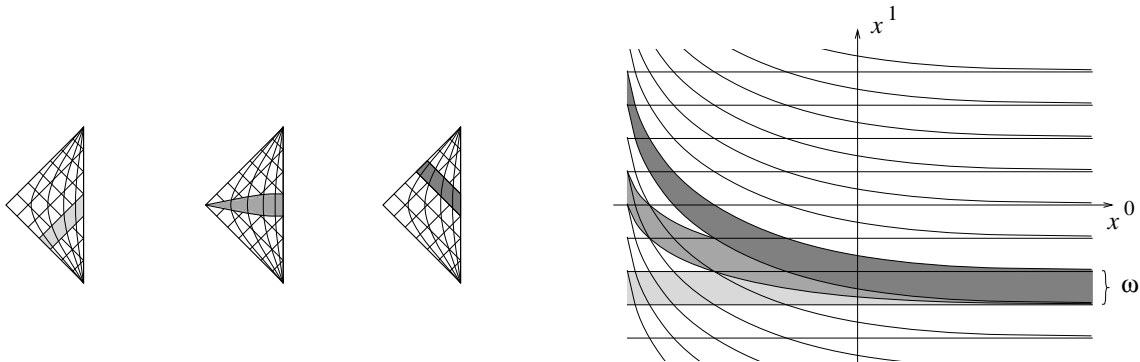
The above theorem is also valid for the case of a restricted symmetry group. For instance, a spacetime is often supposed to have — apart from its metric structure — also an orientation and/or a time-orientation. If the universal covering  $\mathcal{M}$  carries a (time-)orientation and  $\mathcal{H}$  is (time-)orientation-preserving, then also the factor space inherits a (time-)orientation. Now, if two subgroups are conjugate,  $\mathcal{H}' = h\mathcal{H}h^{-1}$ , but the intertwining transformation  $h$  does not preserve the (time-)orientation, then the corresponding factor spaces will have different (time-)orientations (while still being isometric, of course), and should thus be regarded as different. Thus, in this case the conjugacy classes of subgroups should be taken with respect to the restricted symmetry group (e.g.  $\mathcal{G}_+^\uparrow$ ).

The requirement that the subgroup acts freely already rules out some transformations. First of all the subgroup must not contain pure flips at any sector: A flip has a whole line of fixed points, namely an extremal of the kind  $dx^1/dx^0 = -1/h$  (cf. (8) and Fig. 7) as the symmetry axis. This symmetry axis would become a boundary line of the factor space, which then would no longer be maximally extended. Consequently not only  $f$ , but also  $fs_i^n$  and their conjugates have to be omitted.<sup>13</sup>

But also reflexions at a bifurcation point (turning the solution 180° around this bifurcation point, e.g.  $s_1$  in Fig. 8) must be avoided: Not only does the factor space fail to be time-orientable in this case, but also the bifurcation point is a fixed point, which upon factorization develops into a singular ‘conical tip’ making the solution non-smooth. Thus, if the  $k$ th horizon is non-degenerate, then the elements  $s_{k-1}^k = s_{k-1}s_k^{-1}$  and conjugates thereof must not occur in the subgroup  $\mathcal{H}$ .

Let us next assume pure boosts, which form a group isomorphic to  $\mathbb{R}$  ( $\mathbb{R}^{(\text{boost})}$ ). The only discrete subgroups of boosts are the infinite cyclic groups generated by one boost,  $\mathcal{H}_\omega := \{b_\omega^n, n \in \mathbb{Z}\}$ ,  $\omega > 0$ . In the coordinates (2) such a boost  $b_\omega$  is a shift of length  $\omega$  in  $x^1$ -direction. The factor space is then clearly a cylinder. Also, since boosts commute with sector-moves and ‘anticommute’ with flips ( $fb_\omega = b_{-\omega}f$ ), the group  $\mathcal{H}_\omega$  is invariant under conjugation and the parameter  $\omega$  cannot be changed. Thus the cylinders are labelled by

<sup>13</sup>As pointed out previously  $s_i f$  is a flip at sector  $i$ . By the group product one has further  $fs_i^{2k} = s_i^{-k}fs_i^k$  and  $fs_i^{2k-1} = s_i^{-k}(s_i f)s_i^k$ , so they are conjugate to pure flips and thus also pure flips at displaced sectors.



**Figure 10:** A cylinder resulting from pure boosts and possible fundamental regions  $\mathcal{F}$ ; left-hand side in the Penrose diagrams, right-hand side in the EF-chart (2). Boundaries have to be glued along Killing-trajectories, i.e. along the curved lines in the left, resp. along vertical fibres in the right diagram.

one positive real parameter  $\omega$ . In order to find some geometrical meaning of this parameter it is useful to adopt a cut-and-paste approach to the factorization procedure (used already intuitively in Sec. 1): In the above example a fundamental region  $\mathcal{F}$  can be obtained from the patch (2) by cutting out a strip of width  $\omega$  parallel to the  $x^0$ -axis and gluing it together along the frontiers (preserving  $x^0$ , i.e. vertical fibres). Of course there are several choices for  $\mathcal{F}$ ; it need not even be a horizontal strip, but any strip with vertical cross-section  $\omega$  will work (cf. Fig. 10). The width  $\omega$  of this strip (i.e. of the generating shift) is proportional to the length of an  $X^3 = \text{const}$ -path (resp. constant curvature or  $\Phi$ ) running once around the cylinder. This is the desired geometrical observable: for any metric  $g$  (i.e. function  $h$  in (2)) we get a set of distinct solutions parametrized by their ‘size’ or circumference (any positive real number).<sup>14</sup> Let us finally point out that this parameter  $\omega$  has nothing to do with the Casimir-value  $C$ , present in the function  $h$  of the metric. It is a new, *additional* parameter resulting from the factorization.

This all works perfectly well as long as there is only one sector, i.e.,  $h$  has no zeros, but at horizons the boosts do not act properly discontinuous. As a consequence the factor space will not be Hausdorff there. Furthermore, at bifurcation points the action is not free, so the factor space is not even locally homeomorphic to  $\mathbb{R}^2$  there. At the first glance this might seem surprising since when starting from an EF-coordinate patch (2) the construction of Fig. 10 should yield regular cylinders. They are not, however, global: ‘half’ of the extremals approach the (closed) horizon asymptotically, winding around the

<sup>14</sup>In Sec. 1 we took the second continuous parameter besides  $C$  without restriction on its sign. For a comparison with the RPS this is more appropriate, since  $\omega \sim -\omega$  is induced by the *large* diffeomorphism  $x^1 \rightarrow -x^1$ , which is not connected to the identity and thus cannot be generated by the flow of the constraints.

cylinder infinitely often while having only finite length. This phenomenon is well-known from the Taub-NUT space or its two-dimensional analogue as described by Misner [14] (cf. also [10]). A detailed description with illustrations can also be found in [15]. [In **G3**, Fig. 11, for instance, let the EF-coordinates cover the sectors I and II. The above class of incomplete extremals then comprises those which run across the lower horizon from I to IV, leaving the EF-patch. Thus to obtain an extension a second half-cylinder corresponding to IV has to be attached to I via a second copy of the closed horizon and likewise for the sector III, but of course this violates the Hausdorff property.] Similar results hold in general for solutions with zeros of  $h(x^0)$ : whenever the group factored out contains a pure boost we obtain a Taub-NUT like solution (a cylinder-‘bundle’, where at each horizon two sheets meet in a non-Hausdorff manner) labelled by its ‘size’ (metric-induced circumference along a closed  $X^3 = \text{const}$  line). If there are further sector-moves in the group  $\mathcal{H}$ , they have only the effect of identifying different sheets of this cylinder-bundle.

After these preliminaries we will now start a systematic classification of the factor solutions. We do this in order by number and type of horizons and illustrate it by the solutions of the JT-,  $R^2$ -, and KV-model (examples **J1-3**, **R1-5**, and **G1-11**, respectively; for the definition of the models and for all their Penrose diagrams cf. **II**, but also Figs. 2-4, 11-20).

#### No horizons (e.g. **G1,2**, **J3**):

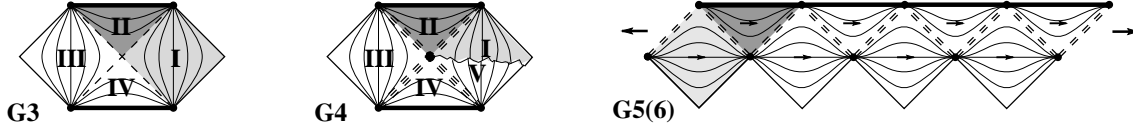
This case has already been covered completely by the above discussion: There is only one sector, so {sector-moves} is trivial. Furthermore, flips are not allowed, thus only boosts remain and they yield cylinders labelled by their circumference (positive real number).

In the following we will continue studying all factorizations possible for solutions *with* horizons. As pointed out before, there exist pathological Taub-NUT-like solutions for all of these cases (resulting from pure boosts). We will from now on exclude them, so pure boosts must not occur in the subgroup  $\mathcal{H}$ . But this also means that no sector-permutation can occur twice with different boost-parameters, since otherwise they could be combined to a non-trivial pure boost. This suggests the following strategy: factor out  $\mathbb{R}^{(\text{boost})}$  from  $\mathcal{G}$ , i.e. project the whole symmetry group onto  $\mathcal{G}/\mathbb{R}^{(\text{boost})} = \mathbb{Z}_2^{(\text{flip})} \ltimes \{\text{sector-moves}\}$ , respectively  $\mathcal{G}_+^\uparrow$  onto  $\mathcal{G}_+^\uparrow/\mathbb{R}^{(\text{boost})} = \{\text{sector-moves}\}^\uparrow$ . All above (non-Taub-NUT) subgroups  $\mathcal{H}$  are mapped one-to-one under this projection. One can thus ‘forget’ the boost-components, and first solve the combinatorial problem of finding the projected subgroups  $\overline{\mathcal{H}}$  (which we will nevertheless simply denote by  $\mathcal{H}$ ). Only afterwards we then deal with re-providing the sector-moves with their boost-parameters.

#### One horizon (e.g. **G3-6**, **R1,2**, **J2**): (Fig. 11)

The group {sector-moves} is in these cases still rather simple:  $\langle s_1 | s_1^2 = 1 \rangle \cong \mathbb{Z}_2$  for a non-degenerate horizon (**G3**, **R1**), and  $\langle s_1 | - \rangle \cong \mathbb{Z}$  for the others. First of all we see that flips have to be omitted altogether: The most general form in which they can occur is  $f s_1^k b_\omega$ , but this is by the group product always conjugate to  $f$  or  $s_1 f$  and thus a pure flip.

The action of {sector-moves} on the manifold is most evident for a zero of even degree, in which case we have the ribbon-structure of **G5(6)**. Then  $s_1$  is a shift of one block to the right (in the situation of Fig. 11). The non-trivial subgroups are the cyclic groups generated by  $s_1^n$ , for some  $n \geq 1$ , and the corresponding factor space is obtained by

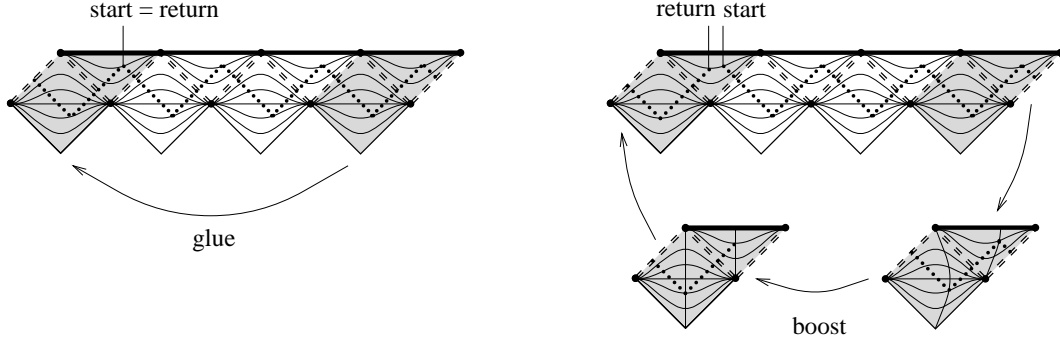


**Figure 11:** Penrose diagrams for one horizon (**G3-6**). For a non-degenerate horizon (**G3**) the vertex point is an (interior) bifurcation point and the manifold a single sheet. For degenerate horizons of odd degree (e.g. **G4**), on the other hand, this point is at an infinite distance, and thus the manifold has infinitely many layers. Finally, for horizons of even degree it is an infinite ribbon (**G5(6)**), interior arrows indicate the Killing field).

identifying blocks which are a multiple of  $n$  blocks apart, i.e., gluing the  $n$ th block onto the zeroth. This yields a cylinder of a ‘circumference’ of  $n$  blocks (see Fig. 12 for  $n = 3$ ). For degenerate horizons of odd degree (e.g. **G4**) the situation is rather similar, only the basic move is now a  $180^\circ$ -rotation around the (infinitely distant) central point, ‘screwing the surface up or down’ (i.e. mapping  $I \rightarrow III$ ,  $II \rightarrow IV$ ,  $III \rightarrow V$ , etc.). Thus we get again cylinders of  $n$  blocks circumference. When passing once around such a cylinder, however, the lightcone tilts upside down  $n$  times, so we have got an  $n$ -*kink*-solution. In particular, the solutions with  $n$  odd are not time-orientable. Finally, for non-deg. horizons (e.g. **G3**) the only non-trivial sector-move is  $s_1$  ( $s_1^2$  is already the identity again), but this is the reflexion at the bifurcation point and thus not a free action. So we get no smooth factor solutions in this case.

If we had restricted ourselves a priori to orientable and time-orientable solutions, then we would have had to start from  $\{\text{sector-moves}\}^\uparrow$ . For **G5(6)** this makes no difference since then  $\{\text{sector-moves}\}^\uparrow = \{\text{sector-moves}\}$ , and indeed all cylinders were time-orientable in this case. For **G4**, on the other hand,  $\{\text{sector-moves}\}^\uparrow$  is a proper subgroup,  $\{\text{sector-moves}\}^\uparrow \cong 2\mathbb{Z} < \mathbb{Z} \cong \{\text{sector-moves}\}$ , generated by  $s_1^2$ . This reflects the fact that only solutions of even kink-number are time-orientable. Finally, for **G3**  $\{\text{sector-moves}\}^\uparrow$  is trivial and there are no factor solutions at all.

We have so far neglected the boost-component of the generator. After all the fully-fledged generator of the subgroup will be  $s_1^n b_\omega$ ! Does this have any consequences on the factor space? According to the theorem at the beginning of this section we must check whether the corresponding subgroups are conjugate. This is not the case here: The generator  $s_1^n b_\omega$  of the subgroup commutes with everything except flips, and even a flip only transforms the group elements to their inverse,  $f s_1^n b_\omega f = s_1^{-n} b_{-\omega} = (s_1^n b_\omega)^{-1}$ . Thus the group remains the same, and one cannot change the boost-parameter  $\omega$  by conjugation. Let us again look for some geometrical meaning of this parameter. In the case of pure boosts discussed above (which also lead to cylinders) we have found the metric-induced circumference as such an observable. This cannot be transferred to the present case, however: there is no closed  $X^3 = \text{const}$ -line along which a circumference could be measured, only the number of blocks  $n$  ‘survives’. So we have to be more inventive: one could, e.g., take a series of null extremals, zigzagging around the cylinder between two fixed values of  $X^3$  (see dotted lines in Fig. 12) and interpret the deviation from being closed,



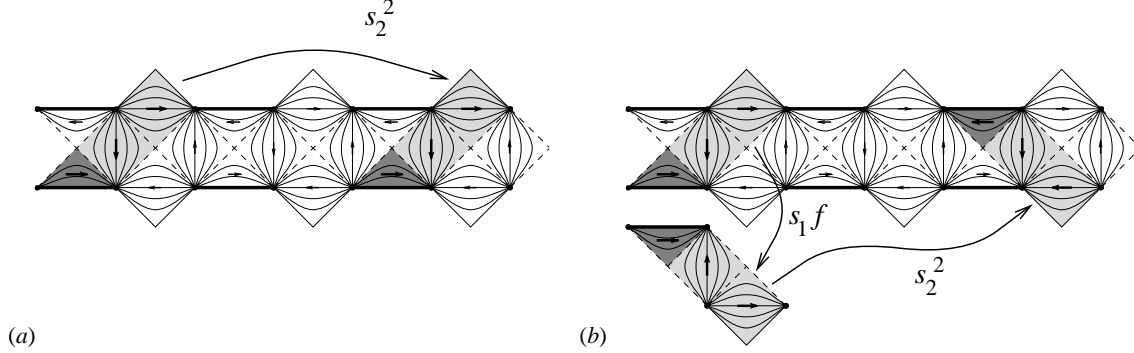
**Figure 12:** Boost-parameter for **G5(6)**. In both cases the generating shift  $s_1^3$  (or its inverse) is used to glue the right-hand shaded patch onto the left-hand one. If this generator has a non-trivial boost-component, then one has to apply a boost before gluing the patches (right-hand figure). To illustrate the effect we have drawn a polygon of null-lines (dotted lines). Due to the boost the endpoint of this null-polygon will be shifted against the start, and this deviation may serve as a measure for the boost-parameter. Of course the same construction can also be applied to the other cases (e.g. **G4**, Fig. 11).

i.e. the distance between starting- and endpoint on this  $X^3 = \text{const}$ -line, as measure for the boost. This distance is of course independent of the choice of the starting point on the  $X^3 = \text{const}$ -line, but it depends on the two  $X^3$ -values; in particular, these  $X^3$ -values can always be chosen such that the deviation is zero (closed polygon). This interpretation of the boost-parameter may be somewhat technical (we will find a much nicer one for e.g. **G8,9** below), but there is certainly no doubt that the parameter is geometrically significant.

### Two non-degenerate horizons (e.g. **G8,9**, **J1**):

Here  $\{\text{sector-moves}\} = \langle s_1, s_2 \mid s_1^2 = 1, (s_1 s_2^{-1})^2 = 1 \rangle$ , where the second relation may be replaced by  $s_1 s_2 s_1 = s_2^{-1}$ . With the help of these two relations any element can be expressed in the form  $s_2^n$  or  $s_2^n s_1$ , and the group can thus (in coincidence with (12)) be written as a semi-direct product,  $\mathbb{Z}_2^{(\text{PT})} \ltimes \mathbb{Z}$ , where  $\mathbb{Z}_2^{(\text{PT})} := \{1, s_1\}$  and  $\mathbb{Z} := \{s_2^n, n \in \mathbb{Z}\} = \{\text{sector-moves}\}^\uparrow$ . However, not all those elements can be permitted:  $s_1$  is a reflexion at one bifurcation point,  $s_2 s_1 = \overset{1}{s_2}$  is a reflexion at the other bifurcation point, and the general element  $s_2^n s_1$  is conjugate to one of them, and thus is also a bifurcation point reflexion. The only freely acting group elements are thus the  $s_2^n$  and they shift the manifold horizontally (in Figs. 13,14) a number of copies sideways. The factor space is then a cylinder. Each cylinder carries again a further real boost-parameter, but let us postpone this discussion and first admit flips (i.e. non-orientable solutions). The most general transformations involving flips (still omitting boosts) are  $s_2^n f$  and  $s_2^n s_1 f$ , but  $s_2^n f$  and  $s_1 f$  are pure flips and cannot be used. A small calculation shows further that the only remaining admissible groups are those generated by *one* element  $s_2^n s_1 f$ ,  $n \neq 0$ . Since  $s_1 f$  is a flip at the middle sector (a reflexion at the horizontal axis in Fig. 13) and  $s_2^n$  is a shift along that axis, the whole transformation is a glide-reflexion and the factor

space a Möbius-strip of  $n$  copies circumference.



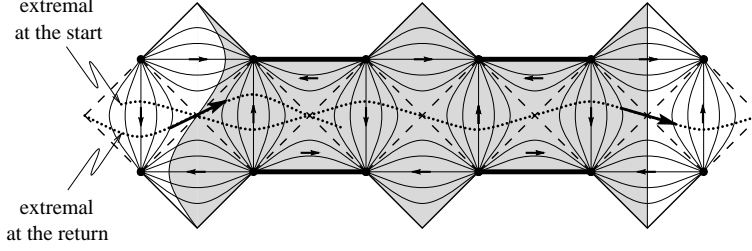
**Figure 13:** Sector-moves for  $G_{8,9}$ ;  $s_2^2$  gives rise to a cylinder (a), while  $s_2^2 s_1 f$  yields a Möbius-strip (b). Note that in (b) the (Killing-)arrows in the blocks identified by  $s_2^2$  do not match.

Now concerning the boost-parameter: The complete cylinder-generator is  $s_2^n b_\omega$ . So we must check whether subgroups with different parameters  $\omega$  are conjugate. Of course conjugation with  $s_2$ ,  $f$ , and boosts does not change the parameter. However, conjugation with  $s_1$  or  $s_1 f$  changes  $\omega$  to its negative,  $(s_1 f)(s_2^n b_\omega)(s_1 f)^{-1} = s_2^n b_{-\omega}$ . Hence the factor spaces corresponding to  $\omega$  and  $-\omega$  are isometric and one could consider to restrict the boost-parameter to non-negative values,  $\omega \in \mathbb{R}_0^+$ . However, the transformation  $s_1 f$  is a flip at the central sector, i.e., a reflexion at the longitudinal axis (horizontal in Figs. 13, 14) and thus inverts the time. Hence, if the spacetime is supposed to have a time-orientation, then the generators with boost-parameter  $\omega$  and  $-\omega$  are no longer conjugate in the restricted symmetry group and the parameter has to range over all of  $\mathbb{R}$ .

In order to give this parameter a geometrical meaning we could of course employ the ‘zigzagging’ null-polygon again, but in this case there is a much better description: In Fig. 14 part of the infinitely extended solution  $G_{8,9}$  is drawn. A generating sector-move shifting the solution two copies (in this example) to the right dictates that the rightmost large patch (four sectors) is pasted onto the corresponding left patch. The boost-parameter then describes that before the pasting a boost has to be applied to the patch. As shown in Fig. 14 its effect is, e.g., that the thin timelike line crossing the right patch vertically has to be glued to the curved line of the left patch. The (shaded) region between these two lines is a possible fundamental region for this factor space! Also, due to the boost the spacelike tangent vector (arrow) to the dotted curve is tilted.

In II it was shown that the bifurcation points are conjugate points and the extremals running between them are those of (8) (in Fig. 14, they have been drawn as dotted curves). They run through the bifurcation points into all directions between the two null-directions. A boost bends them sideways, altering the angle of their tangent. One can now start from a bifurcation point in a certain direction along a spacelike extremal. This extremal will eventually return to the original point, but due to a boost its tangent (cf. arrow in Fig. 14) at the return may be tilted (boosted) against that at the start. This boost is of course independent of the chosen extremal and is thus a true ‘observable’; in particular,





**Figure 14:** Fundamental region (shaded) and interpretation of the boost-parameter for **G8,9**. The right straight boundary has to be glued to the curved left one, and also the extremal (8) (dotted line) returns tilted (i.e. boosted).

there is one solution without a boost. Thus, the cylindrical solutions are parametrized by a positive integer (number of patches) and a real constant parametrizing the boost.<sup>15</sup> As discussed before, if there is no time-orientation, then the boost-parameter has to be restricted to  $\mathbb{R}_0^+$ . In the above interpretation of the boost-parameter this restriction arises if one cannot distinguish between boosts to the past or future. Of course, if a time-orientation is given, then such a distinction is possible. Note that in this case the (time-) direction of the boost is independent of the sense in which the extremal runs through the diagram.

It may seem that for the Möbius-strip one would also have such a continuous parameter due to different flips. However, this is *not* the case: Any two flips are conjugate (via a boost,  $f' \equiv f b_\omega = b_{-\omega/2} f b_{\omega/2} \sim f$ ), hence the corresponding subgroups are conjugate and all Möbius-strips are equivalent. (There is always one extremal (8) which returns unboosted). Also, in contrast with the former examples, a boost cannot be defined consistently on the Möbius-strip; the boost transformation does not ‘factor through’ the canonical projection onto the factor space. This is also seen immediately from Fig. 13 (b), where in the Möbius-case the sectors occasionally have to be identified with their mirror images and thus the arrows indicating the boost-direction do not match. However, locally this Killing symmetry is still present. Hence, the Möbius-strip solution is only parametrized by a positive integer (number of copies).

## 5 More than one generator

So far we have treated the cases where  $\{\text{sector-moves}\}^\dagger \cong \mathbb{Z}$  (one generator) or trivial. In those cases also all subgroups have been one-generator groups  $\mathbb{Z}$ , and the possible topologies have thus been restricted to cylinders and Möbius strips (remember that the subgroup  $\mathcal{H}$  factored out equals the fundamental group of the factor space,  $\pi_1(\mathcal{M}/\mathcal{H})$ ). This situation changes drastically when there is more than one generator, as there are then subgroups of arbitrarily high rank (even infinite).

<sup>15</sup>If we had chosen  $\alpha < 0$  in **(II,13)**, then the whole Penrose diagram would have to be rotated by  $90^\circ$ . The above extremals would then be timelike and the boost at the return could be interpreted nicely as acceleration during one journey around the cylinder.

Ultimately we want to know the conjugacy classes of subgroups of  $\{\text{sector-moves}\}^\uparrow$  (in this section we restrict ourselves to space- and time-orientable solutions; then all those subgroups are properly acting). Subgroups of free groups are again free, so in principle any solution can be obtained by the choice of a free set of generators. But this is only the easier part of the job:

- Given a subgroup (say, in terms of generators) it may be hard to find a *free* set generating this group.
- Also the free generators are by no means unique (one can, e.g., replace  $g_1, g_2, \dots$  by  $g_1, g_2 g_1, \dots$ ). Only the number of free generators (the *rank* of the subgroup) is fixed. So there is the problem to decide whether two sets of generators describe the same group or not.
- We have to combine the subgroups into conjugacy classes.

Since the group is free, these three issues can be solved explicitly (at least for finitely generated subgroups);<sup>16</sup> however, the algorithms are rather cumbersome and thus we will not extend this idea here (details in [13]).

Due to the more complicated fundamental groups it is to be expected that one gets interesting topologies. As already mentioned in Sec. 1, all the solutions will be non-compact (this is also clear, since there is no compact manifold without boundary with a free fundamental group!). Thus it would be nice to have a classification of non-compact surfaces at hand. Unfortunately, however, there is no really satisfactory classification which could be used here (cf. [12]). Let us shortly point out the wealth of different possibilities: A lot of non-compact surfaces can be obtained by cutting holes into compact ones. Of course the number of holes may be infinite, even uncountable (e.g. a Cantor set). A more involved example is that of surfaces of countably infinite genus (number of handles). Finally, there need not even be a countable basis of the topology (this does not happen here, though, since by construction there are only countably many building blocks involved). For an important subcase (finite index), however, the resulting topologies are always of the simple form ‘surface of finite genus with finitely many holes’.

Abstractly the index of a subgroup  $\mathcal{H}$  is the number of cosets of  $\mathcal{H}$ . But it has also a nice geometrical meaning: Since  $\{\text{sector-moves}\}$  acts freely and transitively on the sectors of the same type (and thus on the building blocks), the index of  $\mathcal{H}$  in  $\{\text{sector-moves}\}$  counts the number of building blocks in the fundamental region. Actually, since we started from  $\{\text{sector-moves}\}^\uparrow$ , it would be more convenient to use the index of  $\mathcal{H}$  in that group. If all horizons are of even degree, then  $\{\text{sector-moves}\} = \{\text{sector-moves}\}^\uparrow$  and there is thus no difference. On the other hand, if there are horizons of odd degree, then  $\{\text{sector-moves}\}^\uparrow$  is a subgroup of index 2 of  $\{\text{sector-moves}\}$  and consequently the number of building blocks is *twice* the index of  $\mathcal{H}$  in  $\{\text{sector-moves}\}^\uparrow$ . This is also obvious geometrically, since the fundamental regions in this case are built from patches consisting of two building blocks

---

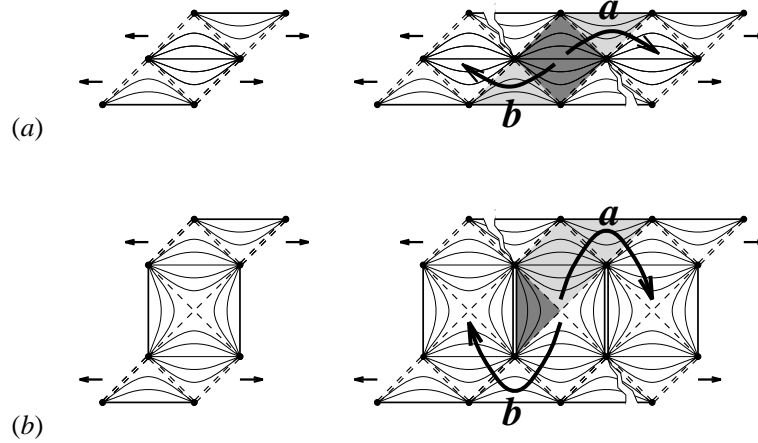
<sup>16</sup>Surprisingly, for non-free groups this is in general impossible. For instance, there is no (general) way to tell whether two given words represent the same group-element (or conjugate elements); and it may also be undecidable whether two presentations describe isomorphic groups (word-, conjugation-, and isomorphism-problem for combinatorial groups, cf. [13]).

(e.g. those situated around a saddle-point in the examples of Figs. 15 (b), 16 below). The index counts the number of these basic patches in the fundamental region then. For finite index, furthermore, it is correlated directly to the rank of  $\mathcal{H}$  via formula (9) (with  $\mathcal{G}$  replaced by  $\{\text{sector-moves}\}^\dagger$ ), thus

$$\text{index } \mathcal{H} = \text{number of basic patches} = \frac{\text{rank } \mathcal{H} - 1}{n - 1} \quad (16)$$

( $n$  being the number of generators of  $\{\text{sector-moves}\}^\dagger$ ).

We will now provide the announced examples, starting with a discussion of the respective combinatorial part of the (orientation and time-orientation-preserving) symmetry group and followed by a discussion of possible factor spaces. Figures 15 and 16 contain basic patches as well as the generators of  $\{\text{sector-moves}\}^\dagger$ . Although this group is the same in all these cases (rank 2), its action for **R5** is different from the others, and correspondingly will be found to give rise to different factor solutions.



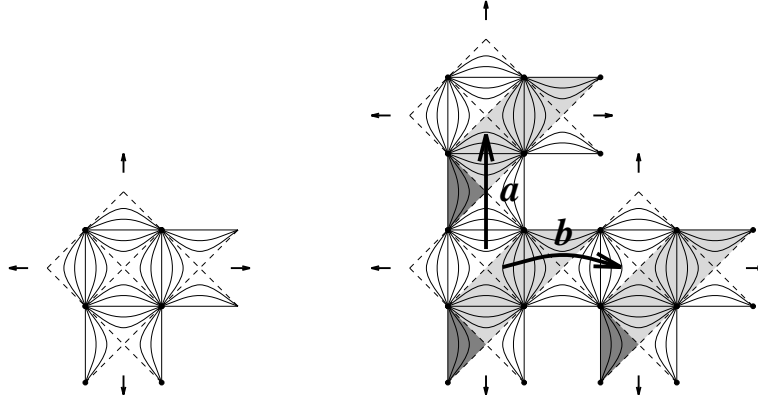
**Figure 15:** Basic patches and generators of  $\{\text{sector-moves}\}^\dagger$  for a fictitious example with two doubly degenerate horizons (a) and for **R3** (b). Note that in the upper example the points in half height are at infinite distance, and that in the lower example the vertical singularities meet, yielding a slit (double lines). Going once around this point/slit leads into a different layer of the universal covering, as indicated by the jagged lines (multilayered Penrose diagrams). Consequently  $ab \neq 1$ .

In the example Fig. 15 (a) (two doubly degenerate horizons) the group  $\{\text{sector-moves}\} (= \{\text{sector-moves}\}^\dagger)$  is free already, with the two generators  $s_1$  and  $s_2$ . Geometrically, however, the moves with basis-sector 1 have a better representation. Let  $a := s_2^{-1} \equiv s_2 s_1^{-1}$  and  $b := s_0^{-1} \equiv s_1^{-1}$ . Clearly  $a$  is a move one block to the right *above* the singularity and  $b$  a move to the left *below* the singularity. Note that since we are in the universal covering their composition,  $ab$ , is *not* the identity but leads into another layer of the covering; if an identification is to be enforced, then the element  $ab$  must occur in the factored out subgroup.

In Fig. 15 (b) only the second horizon is degenerate (**G7,10**, **R3,4**). Thus the basic patch consists of two building blocks. Here  $\{\text{sector-moves}\}^\dagger$  is a proper subgroup of

{sector-moves} and has the two free generators  $s_1s_2$  and  $s_2s_1$  (since the second sector is spatially homogeneous but the basis-sector stationary). Again,  $a := s_2s_1$  is a move one patch to the right *above* the singularity and  $b := s_1s_2$  a move to the left *below* the singularity. Thus the action is similar to that in Fig. 15 (a). However, if (time-)orientation-preservation is not required, then here one has the additional symmetries  $f$  and  $s_1$ , i.e. reflexion at the horizontal axis or at the saddle-point respectively.

Our last example is **R5**, **G11** (three non-degenerate horizons, cf. Figs. 16, 2). Also in this case there are two free generators:  $a := s_2$ , which is a move one patch upwards, and  $b := s_3s_1$ , a move one patch to the right. (The other two potential generators can be expressed in terms of  $a$  and  $b$  by means of the saddle-point relations:  $s_1s_2s_1 = a^{-1}$  and  $s_1s_3 = a^{-1}b^{-1}a$ ). Again, going once around the singularities leads into a new layer of the universal covering ( $\Leftrightarrow [a, b] := aba^{-1}b^{-1} \neq 1$ ).



**Figure 16:** Possible basic patch and generators of  $\{\text{sector-moves}\}^\dagger$  for **R5**. As is seen here, the basic patch need not consist of two *entire* blocks, but the involved sectors may be rearranged somewhat.

Let us now determine the topology for the solutions with finite index. It is clear that then there are also only finitely many boundary segments. [This is a slightly informal terminology, since these ‘boundary segments’ (singularities, null infinities, points at an infinite distance) do *not* belong to the manifold. Still, this can be made precise and such boundaries are called ‘ideal boundaries’ or ‘ends’; we will thus simply use the notion boundary.] The generators of the subgroup  $\mathcal{H}$  determine how the faces of the fundamental region have to be glued and thus also how the boundary segments are put together to form boundary components. This is shown as two examples in Fig. 17. There opposite faces should be glued together, which can be achieved by using the following generators:  $b, a^{-1}ba, a^{-2}ba^2, a^3$  for the left case, and  $b, a^{-1}b^2a, a^{-1}bab^{-1}a, a^2$  in the right case, provided one starts from the lowest basic patch. When starting from another patch, the subgroups and their generators will be conjugates of the above ones, but clearly this does not change the factor solution.

Now, topologically to each boundary component (which is clearly an  $S^1$ ) a disk can be glued. This yields a compact orientable surface, which is completely determined by its

genus. The original manifold is then simply this surface with as many holes as disks had been inserted (each boundary component represents a hole). The genus in turn depends on the rank of the fundamental group  $\pi_1(\mathcal{M}/\mathcal{H}) \cong \mathcal{H}$  and on the number of holes:

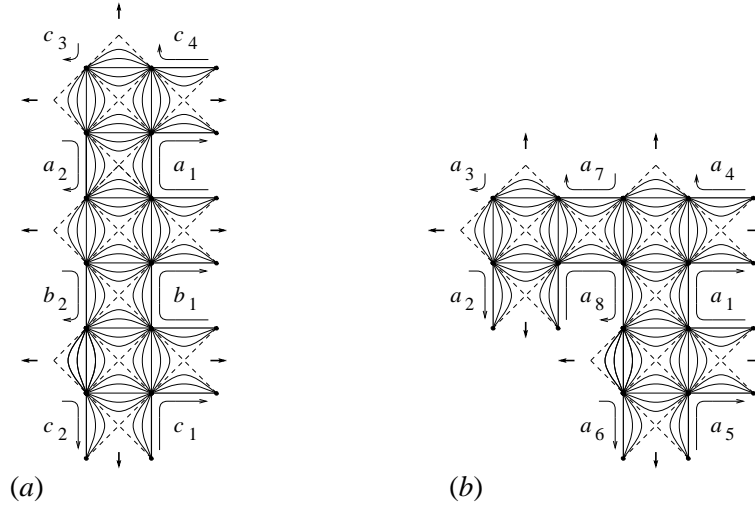
$$\text{rank } \mathcal{H} = \text{rank } \pi_1(\mathcal{M}/\mathcal{H}) = 2 \text{ genus} + (\text{number of holes}) - 1. \quad (17)$$

Expressing the rank by means of (16), this yields

$$\text{genus} = \frac{(\text{number of basic patches}) \cdot (n - 1) - (\text{number of holes})}{2} + 1, \quad (18)$$

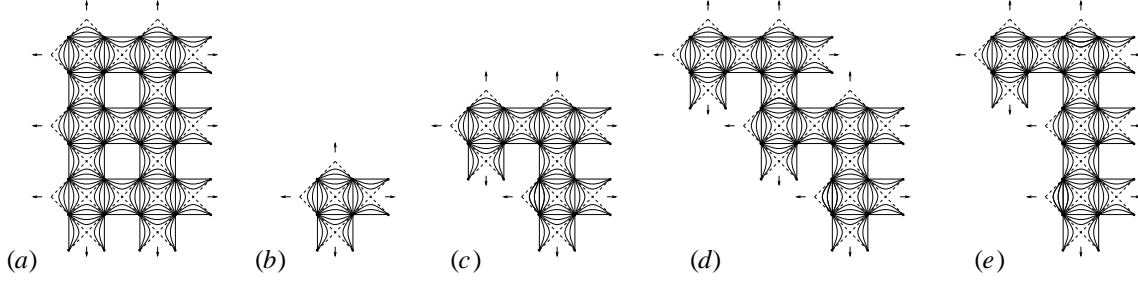
where  $n$  is the number of generators of  $\{\text{sector-moves}\}^\dagger$ .

The general procedure to determine the topologies of the factor spaces can thus be summarized as follows: Draw a fundamental region for your chosen subgroup and determine which faces have to be glued together. Then count the connected boundary components (= number of holes) and calculate the genus from (18). We illustrate this procedure at two examples in Fig. 17. Some further examples for **R5** are given in Fig. 18. Actually,



**Figure 17:** Counting the boundary components (opposite faces have to be glued together). In both cases there are three basic patches (cf. left part of Fig. 16) and thus (use (9) and  $\text{rank } \{\text{sector-moves}\}^\dagger = 2$ !) four generators for  $\mathcal{H}$ , given in the text. However, due to the different number of boundary components (holes) the topologies differ, cf. Eq. (18): In the left example there are three components ( $a_{1,2}$ ,  $b_{1,2}$ ,  $c_{1-4}$ ), thus the topology is that of a torus with three holes. In the right example there is only one component ( $a_{1-8}$ ); therefore this solution is a genus-2-surface with one hole. The resulting manifolds are shown in Fig. 5.

they show that surfaces of any genus ( $\geq 1$ ) and with any number ( $\geq 1$ ) of holes can be obtained: continuing the series Fig. 18 (b), (c), (d) one can increase the number of handles arbitrarily, while attaching single patches from ‘below’ like in (e) allows one to add arbitrarily many holes.



**Figure 18:** Fundamental regions of factor spaces for **R5**. Opposite faces have to be glued together. (a) torus with six holes, (b) torus with one hole, (c) genus-2-surface with hole, (d) genus-3-surface with hole, (e) genus-2-surface with two holes.

For the cases of Fig. 15 (e.g. **R3**) the same analysis can be applied. For instance, it is obvious that cylinders with an arbitrary number of holes ( $\geq 1$ ) can be obtained. Also surfaces of higher genus are possible; however, now the number of holes is always  $\geq 3$  (past and future singularity, and at least one hole in ‘middle height’). [Note that this is no contradiction to the cylinder-with-hole case, since a cylinder with one hole is a sphere (genus-0-surface) with three holes]. In contrast to the **R5**-examples these factor solutions do not have closed timelike curves.

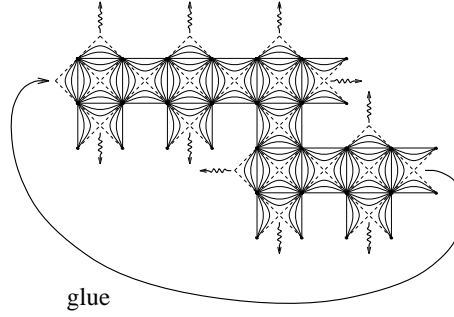
We turn to the cases with infinite index and thus infinite fundamental regions. All subgroups of infinite rank belong to this category, but also many subgroups of finite rank (see below). One topological reason for an infinite rank of the subgroup (and thus also of the fundamental group  $\pi_1$ ) is the occurrence of infinitely many holes. For instance, it was pointed out that in the solution **R3** (Fig. 15) the move  $ab$  is not the identity but leads into a new layer of the universal covering. One can of course enforce the identification of overlapping layers by imposing the relation  $ab \stackrel{!}{=} 1$  and its consequences. This is tantamount to factoring out the group generated by  $ab$  and all its conjugates (the elements  $a^k aba^{-k}$ ,  $k \in \mathbb{Z}$ , form already a free set of generators). The result is a ribbon with infinitely many holes (slits). Clearly the parameter space of such solutions is infinite dimensional now (cf. also remarks at the end of this section). [If in addition one imposed the relation  $a^n \stackrel{!}{=} 1$ , then the previously infinite set of generators would boil down to  $n + 1$  generators ( $a^n$ , and  $a^k aba^{-k}$  for  $0 \leq k < n$ ), and the resulting factor space (finite index again) would be a cylinder with  $n$  holes.]

Likewise, in the example **R5** (Fig. 16) the identification of overlapping layers is obtained by factoring out the infinitely generated commutator subgroup (generated freely e.g. by  $a^m b^n ab^{-n} a^{-m-1}$ ,  $(m, n) \neq (0, 0)$ ; cf. [12]); the factor space is a planar, double-periodic ‘carpet’ then. Adding, furthermore, the generator  $a^n$  (or  $b^n$ ) yields a cylinder with infinitely many holes (e.g. Fig. 18 (a) extended infinitely in vertical direction); and adding both  $a^n$  and  $b^k$  yields a torus with  $nk$  holes, which is again of finite index.

Another possible reason for an infinite rank is an infinite genus (number of handles); such a solution is obtained for instance by continuing the series Fig. 18 (b), (c), (d) infinitely. Of course, both cases can occur simultaneously (infinite number of holes *and*

infinite genus).

Let us now discuss the groups of finite rank and infinite index. Already the universal covering itself, being topologically an open disk (or  $\mathbb{R}^2$ ),<sup>17</sup> provides such an example (with the trivial subgroup factored out). But also one-generator subgroups can by (9) never be of finite index (if  $\text{rank}\{\text{sector-moves}\}^\uparrow \geq 2$ ). These subgroups yield proper (yet slightly pathological) cylinders without holes: Let  $\mathcal{H}_g = \langle g \rangle \equiv \{g^n, n \in \mathbb{Z}\}$  for arbitrary non-trivial  $g \in \{\text{sector-moves}\}^\uparrow$ . In a graphical form this may be thought of as that the generator  $g$  of this infinitely cyclic subgroup<sup>18</sup> defines a path in the universal covering. Now the end-sectors of the path (i.e. of the corresponding ribbon) are identified and at all other junctions the solution is extended infinitely without further identifications (cf. Fig. 19). Thus a topological cylinder, although with a terribly frazzled boundary, is obtained (as before it is possible to smooth out the boundary by a homeomorphism). [The cylinders obtained in this way may have kinks of the lightcone or not. For instance, in the example Fig. 19 the lightcone tilts by (about)  $90^\circ$  and then tilts *back* again. So, this is not a kink in the usual sense of the word; still, there is not one purely spacelike or purely timelike loop on such a cylinder.]



**Figure 19:** Frazzled cylinder from **R5**. Only the utmost left and right faces have to be glued together, such as to make a closed ‘ribbon’ of five patches (a possible generator for this gluing is e.g.  $b^2a^{-1}b^2$ ). At all other faces (indicated  $\rightsquigarrow$ ) the solution has to be extended without further identifications, similarly to the universal covering.

And even for higher ranks of  $\mathcal{H}$  there are solutions of infinite index. The topologies obtained in this way are again of the simple form compact surface with hole(s). There is, however, a much greater flexibility in the rank (which is no longer restricted by formula (16)) as well as in the number of boundary-components (remember that in the examples Fig. 15 all solutions of finite index had at least three boundary-components). Indeed, one can obtain any genus and any nonzero number of holes in this way, as described briefly in

<sup>17</sup>The reader who has difficulties in imagining that such an infinitely branching patch is really homeomorphic to a disk may recall the famous Riemann mapping theorem, which states that any simply connected (proper) open subset of  $\mathbb{R}^2$ , however fractal its frontier might be, is not only homeomorphic but even biholomorphically equivalent to the open unit disk (e.g. [16]). [Clearly, the universal covering is a priori not a subset of the plane (due to the overlapping layers), but by a simple homeomorphism it may be brought into this form.]

<sup>18</sup>or rather the corresponding cyclically reduced element (remember that conjugate groups yield isomorphic factor spaces).

the following paragraph. Eq. (17) is still valid, however, rank  $\mathcal{H}$  can no longer be expressed by (16) but has to be determined directly from the gluings.

As already mentioned we may abuse the classification of compact 2-manifolds with boundary for our purpose; one just has to replace the true boundaries by ‘ideal boundaries’ which do not belong to the manifold. Note that the periphery of the basic patch consists of a couple of faces which have to be glued together pairwise, separated by (‘ideal’) boundary components. [It is topologically immaterial whether these boundary components are pointlike or extended singularities, since they do *not* belong to the manifold; each boundary point can be stretched to an extended segment by a homeomorphism and vice-versa.] The faces come in pairs and the number of pairs equals the rank of the group  $\{\text{sector-moves}\}^\dagger$ . Thus in the present case there are at least two such pairs, which is sufficient to produce fundamental regions with arbitrarily many faces. Furthermore, by virtue of the infinitely branching extensions one can get rid of redundant faces: just extend the solution infinitely at this face so as to obtain a new (‘frazzled’) boundary segment which connects the two adjacent ones, yielding one larger boundary segment. Thus it is possible to produce polygons, the faces of which are to be glued in an arbitrary order. According to [12] this already suffices to produce all topologies announced above.

Finally, we have to discuss the boost-parameters. Since each free generator of  $\mathcal{H}$  carries a boost-parameter, their total number equals the rank  $r$  of this subgroup. Also an interpretation can be given in analogy to the cases dealt with before (zigzagging null-polygon and/or boosted saddle-point extremals). However, not all such choices of an  $r$ -tuple of real numbers are inequivalent. We show this with the example of the torus with three holes (Fig. 17, left part): There are four generators and thus also four boost-parameters, three of them describing the freedom in the horizontal gluing ( $b$ ,  $a^{-1}ba$ , and  $a^{-2}ba^2$ ) and one for the vertical gluing ( $a^3$ ); let us denote them by  $(\omega_1, \omega_2, \omega_3; \omega_4)$ . Now, since conjugate subgroups lead to equivalent factor spaces, we can e.g. conjugate all generators with  $a$ . This leads to new generators, but since  $a$  lies in the normalizer<sup>19</sup>  $\mathcal{NH}$  of  $\mathcal{H}$  in  $\{\text{sector-moves}\}^\dagger$  they still span the same (projected) subgroup  $\mathcal{H}$ . Thus it is possible to express the old generators in terms of the new ones. However, during this procedure the boost-parameters change: For instance, the (full)<sup>20</sup> first generator  $\omega_1 b$  is mapped to  $a^{-1}\omega_1 ba = \omega_1(a^{-1}ba)$ , i.e. the parameter  $\omega_1$  is shifted from the first to the second generator. Altogether, the three ‘horizontal’ boost-parameters  $\omega_{1-3}$  are cyclically permuted and thus we get an equivalence relation among the 4-tuples,  $(\omega_1, \omega_2, \omega_3; \omega_4) \sim (\omega_3, \omega_1, \omega_2; \omega_4)$ . This is of course also geometrically evident: while the timelike loop corresponding to  $\omega_4$  is uniquely characterized, the three spacelike loops corresponding to  $\omega_{1-3}$  are indistinguishable (there is no ‘first one’).

In general, we have a (not necessarily effective) action of the group  $\mathcal{NH}/\mathcal{H}$  on the space of boost-parameters  $\mathbb{R}^r$ . Here  $\mathcal{NH}$  is the normalizer of  $\mathcal{H}$  in  $\{\text{sector-moves}\}^\dagger$  (or, if no (time-)orientation is present, also in  $\{\text{sector-moves}\}$  or  $\mathbb{Z}_2^{(\text{flip})} \ltimes \{\text{sector-moves}\}$ , respectively). The true parameter space is the factor space under this action,  $\mathbb{R}^r/(\mathcal{NH}/\mathcal{H})$ . Locally, it is still  $r$ -dimensional; however, since the action may have fixed points (e.g. in the above example the whole plane  $(\omega, \omega, \omega; \omega_4)$ ), it is an *orbifold* only.

<sup>19</sup>The *normalizer* of a subgroup  $\mathcal{H}$  contains all elements  $g$  for which  $g^{-1}\mathcal{H}g = \mathcal{H}$ .

<sup>20</sup>Here we denoted the boost by  $\omega_i$  instead of  $b_{\omega_i}$  in order to avoid confusion with the sector-move  $b$ .



## 6 Remarks on the constant curvature case

So far we have only dealt with those solutions where the metric (or  $X^3$ -preservation) restricted us to only one Killing field. For reasons of completeness one should treat also the (anti-)deSitter solutions (II,10) of the general model, corresponding to the critical values  $X^3 = X^3_{crit}$  and  $X^a = 0$  (cf. I), which have constant curvature (and zero torsion). Constant curvature manifolds have already occurred as solutions of the Jackiw-Teitelboim (JT) model [17], Eq. (II,11), where, however, the symmetry group was still restricted to only one Killing field since  $X^3$  had to be preserved. Here, on the other hand, these fields are constant all over the spacetime manifold, and thus the solutions have a much higher symmetry.

Already the flat case offers numerous possibilities: The symmetry group (1+1 dimensional Poincaré group) is generated by translations, boosts, and if space- and/or time-orientation need not be preserved, also by spatial and/or time inversion. As before pure reflexions would yield a boundary line (the reflexion axis) or a conical singularity (at the reflexion centre) and boosts a Taub-NUT space. The only fixed-point-free transformations are thus translations and glide-reflexions<sup>21</sup> We have thus the following generators and corresponding factor spaces:

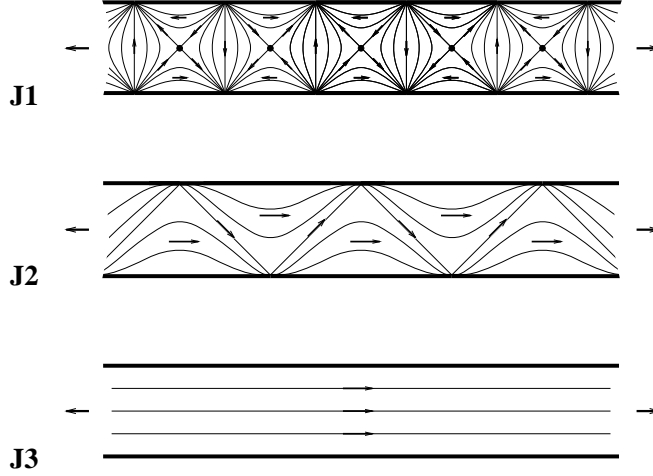
- One translation: Cylinders; parameters = length squared of the generating translation = circumference (squared) of the resulting cylinder (in particular, there is only *one* cylinder with null circumference).
- One glide-reflexion: Möbius-strips, again parametrized by their circumference.
- Two translations: Torus, labelled by three parameters: The lengths (squared) of the two generators  $\vec{a}$  and  $\vec{b}$  and their inner product. Globally, however, this is an overparametrization: Replacing, for instance, the translation vector  $\vec{b}$  by  $\vec{b} + n\vec{a}$  changes one length and the inner product, but still yields the same torus (just the original longitude is now twisted  $n$  times around the torus).
- One translation and one glide-reflexion: Klein bottle. Here only two parameters survive (the inner product of the two generators can be conjugated away always). [This is somewhat similar to the situation cylinders versus Möbius strips in the case of **G8,9** in section 4, where a potential continuous parameter did not occur due to the non-orientability.]

Since two glide-reflexions combine to give a point-reflexion or boost, and three translations generically yield a non-discrete orbit, this exhausts all cases.

Now concerning the ‘proper’ ( $R \neq 0$ ) (anti-)deSitter solutions resp. their universal coverings: Here the situation is slightly more involved. Of course all factor solutions of the JT-model (II,11) are also available for the (anti-)deSitter case, as both have constant curvature and zero torsion. For instance, the Killing fields corresponding to the solutions **J1,2** give rise to cylinders labelled by the block number and a boost-parameter, while **J3** yields cylinders labelled by their (real number) circumference. Now, however, there are

---

<sup>21</sup>A *glide-reflexion* is a translation followed by a reflexion at the (non-null) translation axis. Note that this axis must not be null, if the (orthogonal) reflexion is to be well-defined.



**Figure 20:** Killing fields for the JT-model and deSitter.

three independent Killing fields (e.g. **J1,3** and their Lie-bracket)<sup>22</sup> and thus one would expect further factor solutions. As an example, one has now in addition to cylinders also Möbius-strips of *arbitrary* circumference and not only of an integer number of blocks as in the case of **J1** (**G8,9**). Unfortunately, the full isometry group of the space is  $\tilde{O}(2, 1)$ , whose connected component equals  $\tilde{SL}(2, \mathbb{R})$ , and this group is famous for having no faithful matrix representation and is thus rather difficult to handle. A partial classification has been accomplished by Wolf [11], who deals with homogeneous spaces<sup>23</sup> only and obtains a discrete series of cylinders and Möbius-strips for them. There are strong hints that even in the general case these are the only possible topologies<sup>24</sup> (for instance, it would suffice to show that all properly acting subgroups of  $\tilde{SL}(2, \mathbb{R})$  are isomorphic to  $\mathbb{Z}$ .) However, a proof requires a different approach and might be given elsewhere.

## 7 Non-global inextendible solutions, kinks

The solutions obtained so far have all been geodesically complete, or, if not, the curvature or some physical field blew up at the boundary, rendering a further extension impossible. However, these global spacetimes are not all inextendible ones: it is, for instance, possible that the extremals are all incomplete, the fields and the curvature scalar all remain finite, yet when attempting to extend the solution one runs into problems, because the extension

<sup>22</sup>Note that this is the *only* case with more than one (local) Killing field: Whenever curvature is not constant, the Killing trajectories are restricted to the lines of constant curvature, which leaves at most one independent field. There is thus no 2D-metric with only two local Killing fields.

<sup>23</sup>A space is called *homogeneous*, if its group of isometries acts transitively on it (the space then looks ‘the same’ from every point).

<sup>24</sup>It is clear that no compact topologies can occur: According to [3] compact Lorentz-manifolds should have an Euler characteristic which is zero (i.e., torus or Klein bottle), but by the Gauss-Bonnet theorem this is impossible for non-vanishing constant curvature.

would no longer be smooth or Hausdorff or similar (cf. e.g. the Taub-NUT cylinders of Sec. 4). The purpose of this section shall be to give some further examples, to outline some general features of such solutions, and to discuss to which extent a classification is possible.

A familiar example for the above scenario is the metric [18]

$$g = e^{-2t} (-\cos 2x dt^2 - 2 \sin 2x dt dx + \cos 2x dx^2) , \quad (19)$$

where the coordinate  $x$  is supposed to be periodically wrapped up,  $x \sim x + n\pi$ . These are  $n$ -kink solutions, which means that, loosely speaking, the lightcone tilts upside-down  $n$  times when going along a non-contractible non-selfintersecting loop on the cylindrical spacetime. For  $n = 2$ , however, (19) is nothing but flat Minkowski space, the origin being removed, as is easily seen by introducing polar coordinates

$$\tilde{x} = e^{-t} \cos x , \quad \tilde{t} = e^{-t} \sin x \quad (20)$$

into the metric

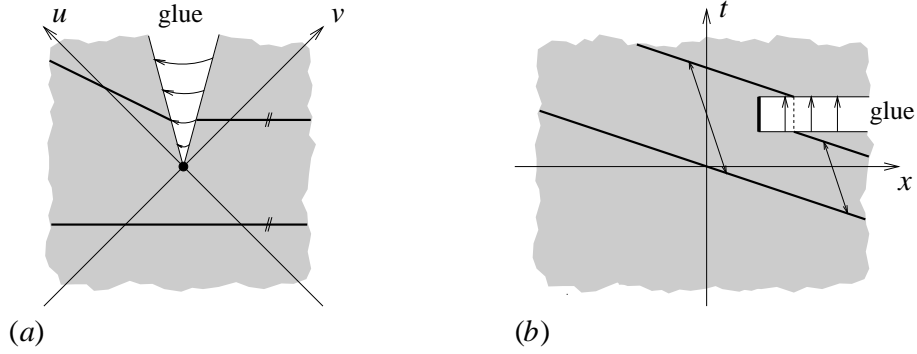
$$g = d\tilde{t}^2 - d\tilde{x}^2 , \quad (21)$$

a fact that seems to have been missed in most of the literature. Consequently, the metric is incomplete at the origin; it has a hole which can easily be filled by inserting a point, leaving ordinary Minkowski space without any kink. For  $n \neq 2$ , on the other hand, (which are covering solutions of the above punctured Minkowski plane, perhaps factored by a point-reflexion) this insertion can no longer be done, because it would yield a ‘branching point’ (conical singularity) at which the extension could not be smooth. Thus these manifolds are inextendible but nevertheless incomplete and certainly the curvature does not diverge anywhere ( $R \equiv 0$ ).

Such a construction is of course possible for *any* spacetime, leading to inextendible  $n \neq 2$ -kinks. However, if there is a Killing symmetry present, then even in the 2-kink situation inextendible metrics can be obtained. To see this let us try to adapt the factorization approach of the previous section to these kink-solutions. First of all, since a point has been removed the manifold is no longer simply connected, so one must pass to its universal covering, which now winds around the removed point infinitely often in new layers (cf. **G3** versus **G4** in Fig. 11). All above kink-solutions can then be obtained by factoring out a ‘rotation’ of a multiple of  $2\pi$  (or  $\pi$ , if there is a point-reflexion symmetry) around the hole. But according to the previous sections there should also occur a kind of boost-parameter. Is it meaningful in this context?

The answer to this question is yes, and this is perhaps best seen at the flat 2-kink example, the Killing field chosen to describe boosts around the (removed) origin. As long as the origin was supposed to belong to the manifold, smoothness singled out one specific boost value for the gluing of the overlapping sectors, leading to Minkowski space; otherwise there would have occurred a conical singularity at the origin. However, if this point is removed, then there is no longer any restriction on the boost-parameter. Its geometrical meaning is that after surrounding the origin a boost has to be applied before gluing or, in terms of fundamental regions, that a wedge has to be removed from the original (punctured) Minkowski space and the resulting edges are glued by the boost (also

the tangents must be mapped with the tangential map of this boost, cf. Fig. 21 (a)). Of course, it is also possible to insert a wedge, but this is equivalent to removing a wedge from an adjacent (stationary) sector. Clearly such a space is everywhere flat (except at the



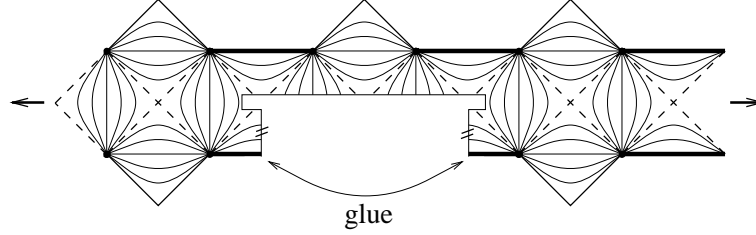
**Figure 21:** (a) Minkowski kink with non-trivial holonomy. This space can be obtained by removing a wedge from flat Minkowski space and gluing together the corresponding boundary lines by a boost. Due to this construction two extremals which are parallel on one side of the origin are mutually boosted on the other side (cf. bold lines). Thus the holonomy is non-trivial (surrounding the origin yields a boosted frame), and at the origin there would occur a conical singularity. (b) Another inextendible Minkowski kink; it has trivial holonomy but the distance of parallels passing the hole changes.

origin, which is considered not to belong to the manifold) but has non-trivial holonomy. For instance, two timelike extremals which are parallel ‘before’ passing the origin at different sides will be mutually boosted afterwards (bold lines in Fig. 21 (a)).

In the above example we chose as Killing symmetry the boosts centered at the origin. However, Minkowski space also exhibits translation symmetries. An analogous construction can be applied also in this case with the following geometrical interpretation: cut out a whole slit (in direction of the chosen translation), remove the strip on one side of the slit, and glue together the resulting faces (cf. Fig. 21 (b)). This manifold has now trivial holonomy, but the metric distance of two generic parallels passing the hole changes. Thus the manifold is so badly distorted that it cannot be completed to ordinary Minkowski space, either. In contrast to the former case this space can still be smoothly extended further: one can simply continue beyond the remaining left edge of the slit (bold line) into an overlapping layer whose upper and lower faces have to be glued together (since the endpoints of the slit have to be identified). This yields a maximally extended cylinder, where the (identified) endpoints of the slit constitute a conical singularity and should be removed.

It is relatively straightforward to write down the metric for the above examples in a circular region around the hole (but not too close to the hole), using smooth but non-analytic functions. Analytic charts are more difficult to obtain, but at least for the case Fig. 21 (b) also this is possible (cf. [19]). This last example, furthermore, can easily be generalized to an arbitrary metric with Killing symmetry: one has just to introduce

Eddington-Finkelstein coordinates (2) in one patch. Then the analogous construction with  $(x, t) \rightarrow (x^0, x^1)$  yields a one-parameter family of inextendible 2-kink solutions (resp.  $2n$ -kink). Explicit charts can be found in [19]. There is (sometimes) even the possibility to introduce discrete parameters like a ‘block-number’: one could, e.g., take **G9** and make a long horizontal slit over a number of blocks, then remove a few blocks on the one side and glue together again (cf. Fig. 22).



**Figure 22:** Yet another kink for **G8,9**.

It is somewhat problematic to give a complete classification of the kink solutions found above. Of course they could be described as factor spaces of *limited* coverings of the original universal covering solution. First, however, this would rather be a mere enumeration of the possible cases than a classification. Secondly, we do not want to distinguish two solutions, one of which is just an extension of the other. Thus we should only consider maximally extended limited coverings. In the above examples they all had only conical singularities, but it is not evident that this should be the most general scenario. Disregarding this question of the extension, the kinks are of course characterized by their kink-number, a (real) boost-parameter, and perhaps further discrete parameters (block-number or the like).

Finally we want to mention that such surgery is not restricted to cylindrical solutions (i.e. one hole only), but, within any of the global solution obtained in the previous sections, one can cut any number of holes, each giving rise to one boost-, one kink-, and perhaps some further discrete parameter. And as is well-known from complex analysis (Riemann surfaces), one can even obtain surfaces of higher genus (e.g. genus 1 with four branching points, etc.) in that way.

## 8 Conclusion

We have succeeded in finding all global (as explained in Sec. 1) maximally extended solutions for generalized dilaton gravity or, more general, for any gravity model with Killing symmetry. The occurring topologies were found to depend only on the number and (degeneracy-)degrees of the Killing horizons within a corresponding Eddington-Finkelstein coordinate patch (2). In particular, for three or more horizons we obtained solutions on non-compact surfaces of arbitrary genus and with an arbitrary nonzero number of holes. Besides these global solutions, where a further extension was impossible due to the

completeness of the extremals or divergent curvature or dilaton fields, we have also found classes of incomplete inextendible solutions, where an extension was impeded by a conical singularity or similar defects (Sec. 7).

As a general rule, the number of additional continuous parameters arising for non-trivial topologies equals the rank of the fundamental group  $\pi_1(M)$ . The dimension of the solution space (including the integration constant  $C$ ) exceeds this rank by one, certainly. In the more general case including Yang-Mills fields, which has not been treated here explicitly, this dimension is  $(\text{rank } \pi_1(M) + 1)(\text{rank (gauge group)} + 1)$ .

In this series the Poisson- $\sigma$ -formalism was used only in the first step, to obtain the local form of the metric (2). Then we proceeded in a purely classical gravitational manner. It seems to be possible, however, to stay all the way within the Poisson- $\sigma$ -framework: In this case the continuous parameter  $C$  as well as the discrete labels encountered above are reobtained as homotopy classes of maps of the 2D world-sheet (spacetime) into symplectic leaves in the target space. The additional continuous (boost) parameters, on the other hand, are found to correspond to generalized “parallel transporters”  $\oint_{\Gamma} A_{\bar{I}}$  around non-contractible loops  $\Gamma$  on the spacetime. We hope to come back to this aspect in more detail elsewhere.

## Acknowledgement

We are grateful to H. Balasin, H.D. Conradi, S.R. Lau, F. Schramm, and V. Schulz for discussions and to W. Kummer for his encouragement during the long-lasting genesis of the present paper. The work has been supported in part by the Austrian Fonds zur Förderung der wissenschaftlichen Forschung (FWF), project P10221-PHY.

## References

- [1] T. Banks and M. O’Loughlin, *Nucl. Phys.* **B362** (1991); 649, S.D. Odintsov and I.L. Shapiro, *Phys. Lett.* **B263** (1991), 183.
- [2] T. Klösch and T. Strobl, *Class. Quantum Grav.* **13** (1996), 965; Corrigendum, *Class. Quantum Grav.* **14** (1997), 825. (Referred to as **I** in the text).
- [3] Y. Choquet-Bruhat, C. DeWitt-Morette, *Analysis, Manifolds and Physics*, North-Holland Physics, 1982.
- [4] T. Klösch and T. Strobl, *Class. Quantum Grav.* **13** (1996), 2395. (Referred to as **II** in the text).
- [5] M.O. Katanaev and I.V. Volovich, *Phys. Lett.* **175B** (1986), 413.
- [6] T. Klösch and T. Strobl, *Classical and Quantum Gravity in 1+1 Dimensions; Part IV: The Quantum Theory*, in preparation.

- [7] T. Strobl, *Phys. Rev.* **D50** (1994), 7346.  
P. Schaller and T. Strobl,  
*Quantization of Field Theories Generalizing Gravity-Yang-Mills Systems on the Cylinder*, in LNP **436**, p. 98, ‘Integrable Models and Strings’, eds. A. Alekseev et al, Springer 1994 or gr-qc/9406027.  
*Mod. Phys. Letts.* **A9** (1994), 3129.  
*Poisson  $\sigma$ -models: A generalization of Gravity-Yang-Mills Systems in Two Dimensions*, in the Proceedings of the International Workshop on ‘Finite Dimensional Integrable Systems’, p. 181-190, Eds. A.N. Sissakian and G.S. Pogosyan, Dubna 1995, or hep-th/9411163  
*Introduction to Poisson  $\sigma$ -Models*, in LNP **469**, p. 321 ‘Low-Dimensional Models in Statistical Physics and Quantum Field Theory’, Eds. H. Grosse and L. Pittner, Springer 1996, or hep-th/9507020.
- [8] A. Ashtekar, *Lectures on Non-Perturbative Canonical Gravity*, World Scientific, Singapore 1991.
- [9] P. Schaller and T. Strobl, *Phys. Lett.* **B337** (1994), 266.
- [10] S.W. Hawking and G.F.R. Ellis, *The Large Scale Structure of Space-Time*, Cambridge University Press, 1973.
- [11] J.A. Wolf, *Spaces of Constant Curvature*, McGraw-Hill, 1967.
- [12] W.S. Massey, *Algebraic Topology: An Introduction*, GTM **56**, Springer, 1977.
- [13] R.C. Lyndon, P.E. Schupp, *Combinatorial Group Theory*, Springer, 1977.
- [14] C.W. Misner in *Relativity Theory and Astrophysics I: Relativity and Cosmology*, ed. J. Ehlers, Lectures in Applied Mathematics, Vol. 8, p. 160, AMS 1967.
- [15] R.P. Geroch, *J. Math. Phys.* **9** (1968), 450.
- [16] R.B. Burckel, *An Introduction to Classical Complex Analysis, Vol. I*, Birkhäuser, 1979.
- [17] B.M. Barbashov, V.V. Nesterenko and A.M. Chervjakov, *Theor. Mat. Phys.* **40** (1979), 15; C. Teitelboim, *Phys. Lett.* **B126** (1983), 41; R. Jackiw, *1984 Quantum Theory of Gravity*, ed S. Christensen (Bristol: Hilger), p. 403.
- [18] K.A. Dunn, T.A. Harriott, J.G. Williams, *J. Math. Phys.* **33** (1992), 1437;  
M. Vasilic, T. Vukasinac, *Class. Quantum Grav.* **13** (1996), 1995.
- [19] T. Klösch and T. Strobl, *A Global View of Kinks in 1+1 Gravity*, preprint, gr-qc/9707053.